An Introduction to the Dynamics of Real and Complex Quadratic Polynomials.

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Abstract

This paper introduces the basic notions of Dynamical systems. We also introduce the Mandelbrot and Julia sets. Both have intricate decorations which make them popular among mathematicians and the public in general, we investigate the geometry of these sets and provide a framework for further analysis.

1 Introduction to Dynamics.

1.1 Basic definitions and concepts.

We aim in this section to introduce the reader to the basic properties of a dynamical system. This paper mostly focuses on complex dynamics, so without a firm grasp of dynamical systems, it would be a futile endeavour to try and understand complex dynamical systems.

Informally, when we say we are studying a dynamical system we are studying the system as it changes through certain periods of time. The study of long term behaviour of such a system could be under iteration (if we are working with a discrete dynamical system) or perhaps as the system changes with time (the continuous case). As alluded to in brackets, we can choose to investigate systems in either a discrete context or a continuous one. Usually it is obvious which one is more natural to choose given a system, indeed it would not make much sense to investigate the changing of weather patterns in blocks of weeks. One can easily see that this is an example of a natural continuous dynamical system, just as say;

$f : \mathbb{N} \longrightarrow \mathbb{N}$ given by $f(x) = x + 2$, with starting point $x = 0,$ is a natural discrete dynamical system. Despite this dichotomy between dynamical systems, the majority of important theorems and results which hold for one type of system, hold for the other. So throughout this paper we will work within the framework of a discrete dynamical system.

Formally, given a non-empty set X, and a function $f : X \longrightarrow X,$ we iterate the function, so we have some starting point $x_0 \in X$, then $f^n(x_0) = \underbrace{f \circ f \circ \ldots \circ f(x_0)}_{n-times}$ for $n \in \mathbb{N}$ describes a dynamical system.

Example 1. Let $X = \mathbb{R}$, $f : X \longrightarrow X$ be given by $f(x) = x^3$.

If we let $x_0 = 2$ be our starting point, $f(x_n) = x_{n+1}$ gives us

$x_0 = 2$, $x_1 = 8$, $x_2 = 512$, $x_3 = 134217728$,...

Definition 2. Given a non empty set X, $f : X \longrightarrow X$. For some $x \in X$

- $\bigcup\limits_{n>0} f^n(x)$ is called the forward orbit of x.
- $\bigcup\limits_{n>0} f^{-n}(x)$ is called the backward orbit of x.

Note that the backward orbit is only defined if f is invertible.

Definition 3. The entire orbit of $x \in X$ is given by $\left\{\bigcup\limits_{n>0} f^n(x)\right\} \bigcup \left\{\bigcup\limits_{n>0} f^{-n}(x)\right\}$.

Our objective when studying a dynamical system is to understand all orbits, ie. for any $x \in X$, and a given function, we want to understand the global nature of it's forward and backward (if they exist!) orbits, since then we will understand the system. In practice this is extremely difficult, indeed later we shall see that even simple looking functions such as the quadratic map $P_c(x) = x^2 + c$, $(x, c \in \mathbb{C}$ or $\mathbb{R})$, display a myriad of complex behaviour, and is still an active topic of research.

Definition 4. Given a non-empty set X, and $f : X \longrightarrow X$.

1. A point $x \in X$ is fixed if $f(x) = x$.
2. A point $x \in X$ is periodic if $f^n(x) = x$ for some $n \in \mathbb{N}$, $n > 0$. The least n such that $x \in X$ is periodic referred to as the prime period of x. Note then, that a fixed point is a periodic point of period one.
3. A point $x \in X$ is preperiodic if $f^k(x)$ is periodic (with $k \in \mathbb{N}$, $k > 0$). Sometimes preperiodic points are called eventually periodic points.

Example 5. Let $X = \mathbb{R}$, $f : X \longrightarrow X$ be given by $f(x) = 3x - 3x^2$.

Then f is fixed when $3x - 3x^2 = x$.

$\therefore 3x - 3x^2 - x = 0.$

$\therefore x(3x - 2) = 0$.

So f has fixed points at $x = 0,\ \frac{2}{3}$.

Now let f be given by $f(x) = x^2 - 3$.

- $x = \frac{1 \pm \sqrt{13}}{2}$ are fixed points of f and so correspond to periodic points of period one.
- $x = 1$ is a periodic point of period 2. Its orbit is given by $x = 1 \longrightarrow -2 \longrightarrow 1 \longrightarrow -2 \longrightarrow ...$
- $x = -2$ is periodic point of period 2. Its orbit is given by $x = -2 \longrightarrow 1 \longrightarrow -2 \longrightarrow 1 \longrightarrow ...$

We found period 2 points by solving $f^2(x) = x$. Note that fixed points will also satisfy $f^2(x) = x$ since if $f(x) = x$, then $f^2(x) = f(f(x)) = f(x) = x$. So we found four periodic points of period 2, with two points of prime period 2.

To find periodic points of period n we must solve $f^n(x) = x$, which will give 2^n points. Clearly as n increases, it becomes impracticle to solve this such an equation using algebraic methods (even a computer will have difficulty computing roots of such magnitude, and even if it succeeds, may give errors). So for a simple function such as $f(x) = 3x - 3x^2$ we cannot fully describe the behaviour orbits by cycling through every n looking for fixed, periodic and preperiodic points. In light of this it is reasonable to find another method for looking at the global structure orbits; one is given in the Section 1.3.

1.2 Periodic Points

Throughout this section we will mostly concern ourselves with the class of polynomials $P_c(z) = z^2 + c$ $(z, c \in \mathbb{C})$ (This function usually serves as a starting point for anyone studying complex dyamics). Whilst we give definitions in complex terms, there is a general crossover with real dynamical systems and $z \in \mathbb{C}$ can easily be replaced with $x \in \mathbb{R}$.

As with systems in the real plane, the goal when studying complex dynamics is to understand the long term behaviour of orbits of points within the domain in question, that is, given $z \in \mathbb{C}$ and a function $f : X \longrightarrow X$ with $X \subseteq \mathbb{C}$, we wish to completely understand, $z, f(z), f^2(z), ...$

The main question we need to answer to understand orbits of a complex system are essentially the same as systems in the real plane:

- What are the fixed points?
- What are the periodic points?
- What are the preperiodic points?
- What is the stable set? (Delayed until Section 2.1)
- What is the behaviour of the critical orbit?

The critical orbit is the forward orbit of a critical point. That is, the forward orbit of a point $z \in \mathbb{C}$ such that $f'(z) = 0$.

Theorem 6. *[AB] Let R be a rational map of degree d, then R has at most $2d - 2$ critical points.* [1]

In fact, if we allocate multiplicity to a critical point, then a rational map R will have exactly $2d - 2$ critical points. It won't become apparent until future sections just how important critical orbits are in the study of complex dynamics, but it is clear that under the assumed importance of critical points, P_c provides a neat starting position as it has just one critical point.

$P_c'(z) = 2z$, hence the critical orbit of P_c is the froward orbit of 0, ie. $0, c, c^2 + c, (c^2 + c) + c, ...$

Another natural question arises, is it possible for a polynomial to have a finite amount of periodic points? The answer is an emphathic no and this is demonstrated in the next Theorem.

Theorem 7. *[BS] A polynomial of degree> 1 has infinitely many periodic points (up to mulitplicity).*

[1] A map is rational if it can be written as the ratio of two polynomials. Of course we require the denominator to be nonzero.

Proof. Clearly as $n \to \infty$ the number of solutions to $f^n(x) - x$ tends to ∞. Indeed, let f had degree $d > 1$, then the degree of $f^n(x) - x$ will be $d^n > 1$, that is, the equation will have d^n solutions, and therefore as $n \to \infty$ so does the number of solutions. Even if it were that the number of periodic points is finite, because the number of solutions to $f^n(x) - x$ tends to ∞, the multiplicity of finite periodic points must tend to ∞.

□

Theorem 8. *[AB] Suppose a polynomial of degree at least two has no periodic points of period k. Then $k = 2$ and the polynomial is conjugate to $z^2 - z$.*

Theorem 9. *If P_c has an attracting periodic orbit, then the critical orbit is attracted to it.*

These three Theorems show that a polynomial of degree at least two will have infinitely many periodic points, but at most one of them can be attracting. In fact we have an even stronger result, given in this paper for completeness:

Theorem 10. *[MS] The total number of attracting, super-attracting, neutral points of a rational map of degree d, is at most $2d - 2$.*

Generally given any $z \in \mathbb{C}$, we wish to know if $P_c^n(z) \to \infty$ as $n \to \infty$, this gives us a rich source of information about the dynamics of P_c and will motivate a lot of this section. With this in mind we give the next definition:

Definition 11. Let $\lambda = |P_c'(z)|$, where z is a fixed point (Note that at $z = \infty$, this isn't defined. We work around this by setting $\lambda = \frac{1}{|P_c'(z)|}$ at $z = \infty$). Then if:

- $\lambda = 0$; z is called a super-attracting fixed point.
- $0 < \lambda < 1$; z is called an attracting fixed point.
- $\lambda = 1$; z is called a neutral fixed point.
- $\lambda > 1$; z is called a repelling fixed point.

Similarly we can replace periodic for fixed in the definition, if z is a periodic point of period n then we let $\lambda = |(P_c^n)'(z)|$.

Armed with the notion of an attracting periodic orbit, the logical question arises; What points are attracted to this attracting orbit?

Definition 12. Given an attracting fixed point $z_0 \in \mathbb{C}$, we define its basin of attraction, denoted $A(z_0)$, to be the set of points which converge to z_0 under iteration. That is, the $z \in \mathbb{C}$ such that $f^n(z) \to z_0$ as $n \to \infty$.

We may define the basin of attraction of an attracting periodic orbit:

Suppose $\{z_0, ..., z_k\}$ is an attracting periodic orbit of a function f.

Then;

$f^{k+1}(z_0) = z_0$

$f^{k+1}(z_1) = f^k(z_2) = ... = f(z_0) = z_1$

$\vdots$

$f^{k+1}(z_k) = z_k$

So each $z_i \in \mathbb{C}$, $0 \leqslant i \leqslant k$, is fixed for f^{k+1}. The the basin of attraction of $\{z_0, ..., z_k\}$ is the union of the basin's of attraction of z_i under f^{k+1}.

IE. $A(\{z_0, ..., z_k\}) = \bigcup_{i=0}^{k} A(z_i)$

The immediate basin of attraction is the connected component of $A(z_0)$ containing z_0.

Similarly the immediate basin of attraction of an orbit is the union of the connected components of $A(z_i)$ containing z_i.

One might wonder if the immediate basin of attraction and the basin of attraction are not one and the same, this turns out not to be true; in general the immediate basin of attraction is smaller than $A(z)$; sometimes $A(z)$ contains infinitely many components.

Proposition 13. *$A(z)$ is open. (z fixed)*

Proof. Let z_0 be a fixed point of an analytic function $f : \mathbb{C} \to \mathbb{C}$ and let $A^{'}(z_0)$ denote it's immediate basin of attraction. [2]

First we must show that given some z in a neighbourhood U of z_0, then $f^n(z) \to z_0$ as $n \to \infty$ IE. $z \in A(z_0)$.

[2]A complex function f is said to be analytic in some subset $U \subset \mathbb{C}$ if it is differentiable at each point in U and is single valued.

z_0 is an attracting fixed point, so $\lambda < 1$. Therefore $\exists k < 1$ such that $\lambda < k < 1$.

Then $|f(z) - z_0| < k|z - z_0|$, $\Rightarrow |f^n(z) - z_0| < k^n|z - z_0| \to 0$ as $n \to \infty$.

Therefore z converges to z_0, so we have $z \in A'(z_0)$.

In light of this, it easy to see that the basin of attraction of z_0 is in fact the union of backward iterates of open neighborhoods of z_0.

That is, $A(z_0) = \bigcup_{n>0} f^{-n}(U)$.

So $A(z_0)$ is the union of backward orbits of points which converge to z_0 within an open neighbourhood, and hence is open.

□

This technique sheds light on why we require $\lambda < 1$ for attracting points, likewise it becomes clear why we require $\lambda > 1$ for repelling points:

If $\lambda > 1$, then $\exists k > 1$ such that $\lambda > k > 1$. Let z_0 be a repelling fixed point.

Suppose (for contradiction) $z \to z_0$ under iterations, then $|f(z) - z_0| < |z - z_0|$, but we have $|f(z) - z_0| > k|z - z_0|$.

$\Rightarrow |f^n(z) - z_0| > k^n|z - z_0| \to \infty$ as $n \to \infty$. So $f^n(z)$ doesn't converge to z_0.

Proposition 14. *The immediate basin of attraction of a (finite) attracting periodic point is simply connected.*

In general the basin of attraction consists of many (possibly infinite) components, whereas the immediate basin of attraction is the set containing the attractive point.

Example 15. Consider the polynomial $P_{-2}(z) = z^2 - 2$. We immediately see that $z = \infty$ is a super-attracting fixed point of P_{-2}.

What is $A(\infty)$?

To see this, a change of co-ordinates, whilst not essential, helps a great deal.

Let $f(z) = z + \frac{1}{z}$ be defined on $\{z \mid |z| > 1\}$. Then $P_{-2}(f(z)) = (f(z))^2 - 2 = (z + \frac{1}{z})^2 - 2 = z^2 + 2 + \frac{1}{z^2} - 2 = z^2 + \frac{1}{z^2} = f(z^2)$.

$\Rightarrow f^{-1} \circ P_{-2} \circ f(z) = z^2$.

Now this gives a conjugacy between P_{-2} defined on $\overline{\mathbb{C}} \setminus [-2, 2]$ and z^2 on $\{z \mid |z| > 1\}$. [3]

It is easy to see that f maps the unit disk onto $[-2, 2]$.

Let $|z| \leq 1$, then $|f(z)| = |z + \frac{1}{z}| \leq |z| + |\frac{1}{z}| \leq 1 + 1 = 2$. Hence $f(z) \in [-2, 2]$.

So we see that f maps $\{z \mid |z| > 1\}$ to $\overline{\mathbb{C}} \setminus [-2, 2]$.

Hence for any $z \in \overline{\mathbb{C}} \setminus [-2, 2]$, $P_{-2}(z) \to \infty$ under iteration. Thus $A(\infty) = \overline{\mathbb{C}} \setminus [-2, 2]$.

1.2.1 Topological Conjugacy

As we have seen (and shall see in subsequent sections), the ability to conjugate seemingly complicated functions to "simpler" functions in order to study the dynamical properties of said function is a useful tool. We formalise the notion of conjugacy in this section:

Definition 16. Let X, Y be two non-empty sets, and $f : X \to Y$, $g : Y \to X$. f and g are said to be topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ f(x) = g \circ h(x)$ for every $x \in X$.[4]

If f and g are topologically conjugate then the dynamics they impose upon X and Y are equivalent to one another:

[3] We are working in $\overline{\mathbb{C}}$, the one point compactification of $\mathbb{C}$. Without this, the abstract point ∞ would not be defined and so would not be a fixed point of the polynomial.

[4] A function $h : X \to Y$ is a homeomorphism if h is bijective, and both h, h^{-1} are continuous.

Let $(X,\ f : X \to Y)$ define a dynamical system and let x_0 be our starting point, so the orbit of x_0 is as follows:

$x_0, f(x_0), f(f(x_0)), \ldots$ then under h, we have $h(x_i) = y_i$.

So $y_0 = x_0, g(y_0) = h(f(x_0)) = h(x_1) = y_1, \ldots$

IE. Under h the two orbits are equivalent.

Note then that $g^n(h(x_0) = h(f^n(x_0))$.

1.2.2 Periodic points continued

Ideally we would like to be able to conjugate an analytic function near any fixed or periodic point, be it super-attracting, attracting, neutral or repelling. That is, we would like to be able to find a “simple” function (hopefully linear or quadratic) which is conjugate to the function in question. Theorem 8 is incredibly powerful, it tells us that if a polynomial isn’t conjugate to $z^2 - z$, then periodic cycles aren’t absent from the polynomial.

This section deals with Theorems involving conjugates near attracting fixed points. There are results for the other types of fixed points, we omit them and focus completely on strictly attracting fixed points. Of course this can be extended to periodic points by using the property; z is a periodic point (of period n) of a polynomial R if and only if R^n fixes z.

The Theorem for linearisation near attracting fixed points was first presented by the mathematician Koenigs (1884).

Theorem 17. *Suppose that a polynomial f has the properties:*

$f(0) = 0$ *(we could have chosen any fixed point but by a translation,* 0 *will do)*

$f'(0) = \alpha$ *with* $\alpha \neq 0$ *and* $|\alpha| < 1$

f *is analytic in some neighbourhood of the origin.*

Then there exists an analytic function ϕ*, defined on a neighbourhood of* 0*, with properties:*

$\phi(0) = 0$

$\phi'(0) = 1$

$\phi^{-1} \circ f \circ \phi(z) = \alpha z.$

Proof. Let U be the neighbourhood of 0, for which f is analytic.

Define $\phi_n(z) = \frac{f^n(z)}{\alpha^n}$ and let $\phi(z) = \lim\limits_{n\to\infty} \frac{f^n(z)}{\alpha^n}$.

$\phi_n \circ f(z) = \phi_n(f(z)) = \frac{f^{n+1}(z)}{\alpha^n} = \frac{\alpha f^{n+1}(z)}{\alpha^{n+1}} = \alpha\phi_{n+1}(z)$

Hence $\phi_n \circ f = \alpha\phi_{n+1}$. If ϕ_n converges uniformly to ϕ then we have the function equation $\phi \circ f(z) = \alpha\phi(z) \Longrightarrow \phi \circ f \circ \phi^{-1}(z) = \alpha z$.

This is dependent on the convergence of ϕ_n to ϕ.

Note $\exists \delta > 0$ such that for $z \in B_\delta(0) \subseteq U$. Then $|f(z) - \alpha z| \leq c|z|^2$ for some c. (1)

$|f(z)| \leq |\alpha z| + c|z|^2 \leqslant |\alpha||z| + c|z|^2 = |z|(|\alpha| + c|z|) \leqslant |z|(|\alpha| + c\delta)$ since $|z| \leqslant \delta$.

Let $|k| = |\alpha| + c\delta$, note that if we choose δ small enough then for the c in (1). We have $|\alpha| < |k| < 1$. Also note that $|\alpha| < |k|^2 < |k| < 1$.

So $|f(z)| \leqslant k|z|$. Hence $|f^n(z)| \leqslant k^n|z|$. (2)

Now $\mid \phi_{n+1}(z) - \phi_n(z) \mid = \mid \frac{f^{n+1}(z)}{\alpha^{n+1}} - \frac{f^n(z)}{\alpha^n} \mid = \mid \frac{f^n(f(z)) - \alpha f^n(z)}{\alpha^{n+1}} \mid = \frac{1}{\alpha^{n+1}} \mid f^n(f(z)) - \alpha f^n(z) \mid \underset{\text{by } (1)}{\leq} \frac{1}{\alpha^{n+1}}.c \mid f^n(z) \mid^2 \underset{\text{by } (2)}{\leq} \frac{1}{\alpha^{n+1}}.ck^{2n} \mid z \mid$ Because $|\alpha| < |k|^2 < |k| < 1$, as we compute $\mid \phi_{n+j+1}(z) - \phi_{n+j}(z) \mid (j \in \mathbb{N})$ it is clear that the bound $\frac{1}{\alpha^{n+1}}ck^{2n}$ will decrease at an exponential rate. Thus for an arbitrary $m \in \mathbb{N}$, the summation $\sum\limits_{0 \leqslant j \leqslant m} \phi_{n+j+1} - \phi_{n+j}$ converges. This summation is equal to $\phi_{n+m+1} - \phi_n$, and so we have convergence for any arbitrary m.

So ϕ converges uniformly for $z \in B_\delta(0) \subseteq U$.

Now we prove $\phi'(0) = \lim_{n\to\infty} \frac{d}{dz}(\frac{f^n(z)}{\alpha^n}) \mid_{z=0} = 1$.

Since $(f^n)'(z) = (f^{n-1} \circ f)'(z)$, we can apply the chain rule: $(f^n)'(z) = (f^{n-1})'(f(z)).f'(z)$.

But $f^{n-1} = f^{n-2} \circ f$. So continuing in this way we get $(f^n)'(z) = \underbrace{f'(f^n(z)).f'(z) \ldots f'(z)}_{n\ times}$.

Using the relations, $f(z) = z \implies f^n(z) = z$, $f(0) = 0$, $f'(0) = \alpha$. We get $(f^n)'(z) \mid_{z=0} = \alpha^n$.

Thus $\phi'(0) = \lim_{n\to\infty} \frac{d}{dz}(\frac{f^n(z)}{\alpha^n}) \mid_{z=0} = \frac{\alpha^n}{\alpha^n} = 1$

□

So we have successfully shown that the behaviour of polynomials near attracting fixed points are relatively simple, and the polynomial in question can be conjugated to a linear map. In light of this, the dynamics near attracting fixed points are very well understood. We omit more in depth study of periodic points and focus now on a convenient method for analysing dynamical systems.

1.3 Graphical Analysis & Bifurcation Theory

Graphical Analysis of functions (in this case cobweb diagrams) allows us to see the global structure of a system in pictorial form. As we iterate a function (such as a quadratic polynomial) the degree of the function increases rapidly. Because of this, it becomes near impossible to effectively analyse the dynamical structure of said function with numerical techniques alone. This is where graphical analysis becomes a great help. Cobweb diagrams show us fixed and periodic points clearly, and it is easy to see from the diagrams what type of fixed or periodic point the function contains. IE. Is it repelling, neutral or attractive?

It also shows us bifurcations of a system, which is discussed later in the chapter.

To this end, we give an illustrative example: [D4]

Let $F(x) = x^2$

We then draw alongside F the function $g(x) = x$:

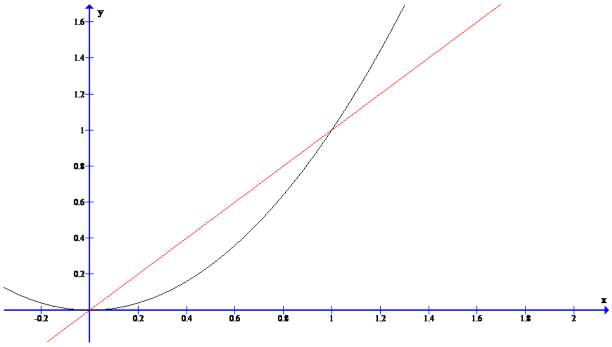


Fig $1.3.1$

Suppose then that we want to investigate the iterates of some $x_0 \in \mathbb{R}$.

We mark off (x_0, x_0) on Fig $1.3.1$ and draw a straight line to $F(x)$, and mark off $(x_0, F(x_0))$.

To investigate the second iterate we draw a straight line from $(x_0, F(x_0))$ to $(F(x_0), F(x_0))$ on $g(x)$, and then proceed to draw a straight line from $g(x)$ to $F(x)$, giving $(F(x_0), F^2(x_0))$. repeating this process we can find the n^{th} iterate of x_0 by finding $(F^{n-1}(x_0), F^n(x_0))$.

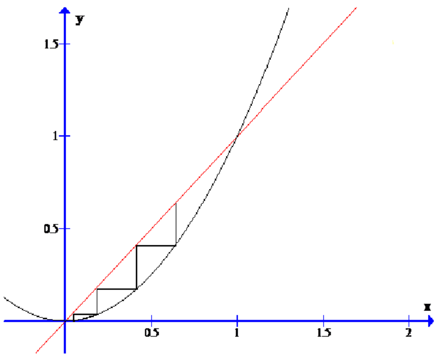


Fig $1.3.2$

As mentioned earlier, this technique is especially useful for deciding the nature of fixed points.

From Fig $1.3.2$, we see $x_0 = 0$ is an attracting fixed point, and $x_1 = 1$ is a repelling fixed point.

Example 18. A bifurcation in a dynamical system refers to changes to the nature of the system, as the parameters effecting the system change, This section aims to give an introduction to bifurcation theory and how it plays a pivotal role in the study of dynamical systems. This is perhaps best illustrated by an example, $P_c(x) = x^2 + c$, we show how the dynamics of P_c change as c varies.[D3]

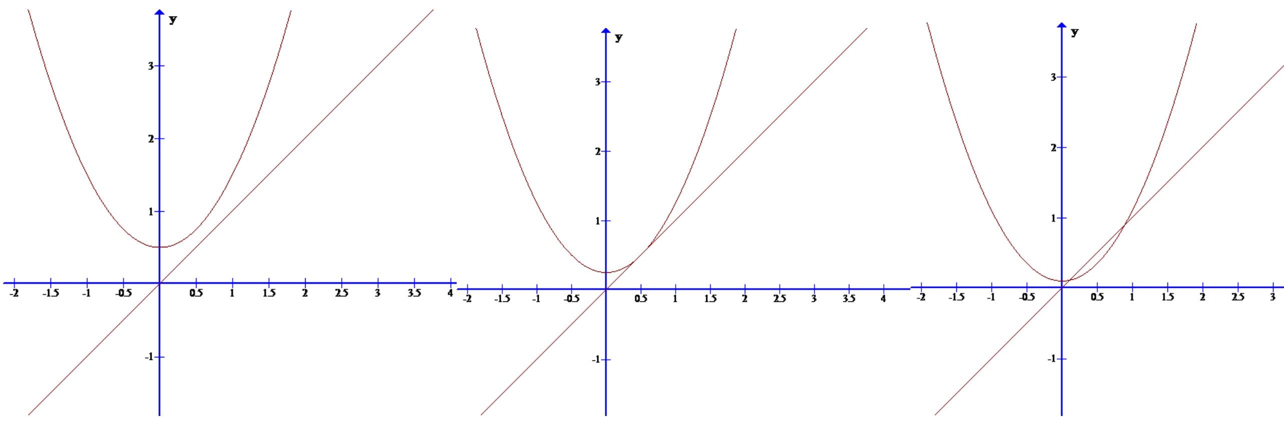
- P_c is fixed when $P_c(x) = x \iff x^2 + c = x \iff x^2 - x + c = 0$

So P_c is fixed for $x = \frac{1 \pm \sqrt{1-4c}}{2}$

Let us denote these two fixed points by $p_1 = \frac{1+\sqrt{1-4c}}{2}$ and $p_2 = \frac{1-\sqrt{1-4c}}{2}$.

Note now that when:

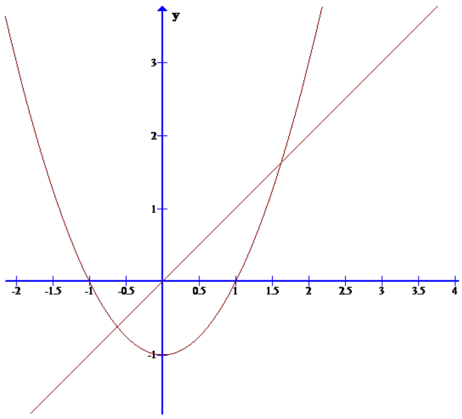
- $c > \frac{1}{4}$ both p_1, p_2 are not real, and so P_c has no fixed points in the real plane.
- $c = \frac{1}{4}$. $p_1 = p_2$ and P_c has just one fixed point of multiplicity 2, at $x = \frac{1}{2}$.
- $c < \frac{1}{4}$, then $1 - 4c > 0$ so $p_1 > p_2$ and P_c has two fixed points in the real plane.



Graph of P_c for $c < \frac{1}{4}$, $c = \frac{1}{4}$ and $c > \frac{1}{4}$

From graphical analysis we can see one of the fixed points is attracting, and the other repelling.

For example take $c = -1$.

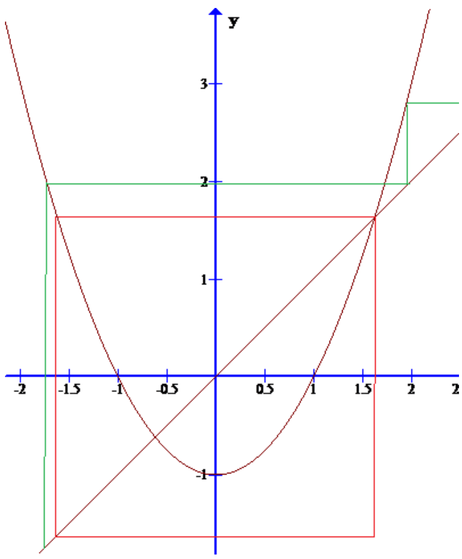


Graph of P_c for $c = -1$

Let us look at $x = \frac{5}{3}$, $P_c(\frac{5}{3}) = \frac{16}{9}, \ldots$

We can see that for any $x = p_1 + \epsilon$ $(\epsilon > 0)$, the orbit of x tends to ∞.

Hence p_1 is an repelling fixed point, whilst p_2 is an attracting fixed point, IE. small perturbations around p_2 do not greatly effect the dynamics whilst for p_1 they do. This gives a very simple example of a bifurcation within P_c, as c varies P_c has no fixed points, then one, then two. The sudden "birth" of two fixed points such as in this case is commonly referred to as a saddle-node bifurcation. The question then becomes, for what values $x, c \in \mathbb{R}$ do we look at in order to study bifurcations within P_c?



Graph of P_c for $c = -1$

If $|x| > p_1$ then $P_c^n(x) \to \infty$ as $x \to \infty$. By the above, we know p_1 is repelling, and this is the "largest" fixed point.

So we may restrict our attention to $x \in [-p_1, p_1]$. Note that for $c \in [-2, \frac{1}{4}]$ we always have $p_2 \in (-p_1, p_1)$:

$-\sqrt{1-4c} < \sqrt{1-4c} \Longrightarrow 1 - \sqrt{1-4c} < 1 + \sqrt{1-4c}$.

$\Longrightarrow \frac{1-\sqrt{1-4c}}{2} < \frac{1+\sqrt{1-4c}}{2}$.

Therefore, $p_2 < p_1$, but $-1 < 1 \Longrightarrow \frac{1-\sqrt{1-4c}}{2} < \frac{1+\sqrt{1-4c}}{2}$.

So $-p_1 < p_2$.

Hence $-p_1 < p_2 < p_1$.

(Note that for $c > \frac{1}{4}$ then $P_c^n(x) \to \infty$ as $n \to \infty$. We shall later see that when $c \leq -2$, the quadratic map P_c is chaotic, and so orbits that don't tend to infinity do exist but finding them is rather complicated. For this reason, we shall only concern ourselves for the moment with the class of maps P_c with $c \in [-2, \frac{1}{4}]$)

By the above, if we restrict our attention to $x \in [-p_1, p_1]$ we will never exclude the possibility $x = p_2$ IE. We will never "miss" the other fixed point. Hence we shall only consider orbits for $x \in [-p_1, p_1]$; the orbits for which we see bifurcations. From future analysis of the Mandelbrot set along the real line, orbits in $[-p_1, p_1]$ are attracted to an attracting fixed point for $\frac{-3}{4} \leq x \leq \frac{1}{4}$, and orbits in $[-p_1, p_1]$ are attracted to a periodic orbit of period 2 for $\frac{-5}{4} \leq c \leq \frac{-3}{4}$.

For $c \in [\frac{-3}{4}, \frac{1}{4}]$, orbits in $[-p_1, p_1]$ are attracted to a fixed point, but we have seen that p_2 is an attracting fixed point, so orbits are attracted to p_2, IE. the basin of attraction for p_2 is $(-p_1, p_1)$.

As c decreases through $\frac{-3}{4}$, we see orbits in $(-p_1, p_1)$ go from being attracted to a fixed point to being attracted to a period 2 orbit. This gives a simple example of a bifurcation in P_c.

It might be reasonable to assume that under this bifurcation we would lose a fixed point. This turns out not to be true:

- For $c \in (\frac{-3}{4}, \frac{1}{4})$; $\mid P_c'(p_2) \mid = \mid 2p_2 \mid = \mid 2(\frac{1-\sqrt{1-4c}}{2}) \mid = \mid 1 - \sqrt{1-4c} \mid < 1$, so p_2 is attracting as show by graphical analysis.
- For $c = \frac{-3}{4}$; $\mid P_c'(p_2) \mid = \mid 2p_2 \mid = \mid 2(\frac{1-\sqrt{1-4.\frac{-3}{4}}}{2}) \mid = \mid 1 - \sqrt{4} \mid = \mid -1 \mid = 1$, so for $c = \frac{-3}{4}$, p_2 is a neutral fixed point.
- For $c \in (\frac{-5}{4}, \frac{-3}{4})$ (In fact for $c < \frac{-3}{4}$); $\mid P_c'(p_2) \mid = \mid 2p_2 \mid = \mid 2(\frac{1-\sqrt{1-4c}}{2}) \mid > 1$, so p_2 is a repelling fixed point.

When p_2 becomes repelling we see a new attracting period 2 orbit is born:

$P_c^2(x) = (x^2 + c)^2 + c = x$, which gives;

$\therefore\ x^4 + 2cx^2 - x + c^2 + c = 0$

$\therefore\ (x^2 - x + c)(x^2 + x + c + 1) = 0$

Therefore $x = p_1, p_2$ and,

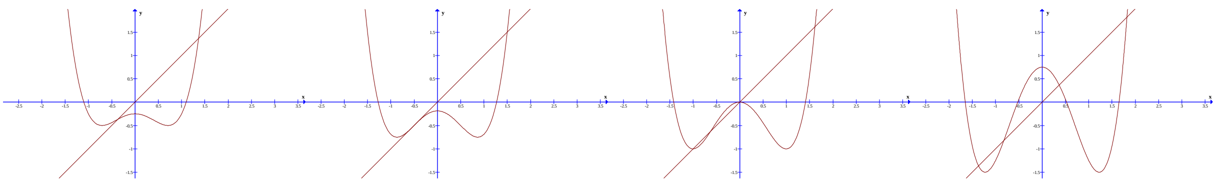
$x = \frac{1 \pm \sqrt{1-4(c+1)}}{2} = \frac{1 \pm \sqrt{-4c-3}}{2}$, two fixed points of P_c^2, so the new attracting period 2 orbit.

We already know two of the four roots to this polynomial since a fixed point of P_c is also a fixed point of P_c^2 .

(If $P_c(x) = x$ then $P_c^2(x) = P_c(P_c(x)) = P_c(x) = x$. More generally if x is fixed for P_c^n, then it is fixed for P_c^{n+1} ($n \in \mathbb{N}$), suppose $P_c^n(x) = x$, then $P_c^{n+1}(x) = P_c(P_c^n(x)) = P_c(x) = x$)

For higher period cycles though we have to solve a polynomial of degree 2^n, whilst only knowing $\frac{2^n}{2}$ roots before hand. As mentioned earlier, solving this is a hopeless task for large n.

So when studying the bifurcations of periodic points for P_c as c varies, it is most common to use graphical analysis.



Graphs for P_c^2 when $c < \frac{-3}{4}$, $c = \frac{-3}{4}$, $\frac{-5}{4} < c < \frac{-3}{4}$ and $c < \frac{-5}{4}$

As c decreases through $\frac{-5}{4}$ the fixed points for P_c^2 (period 2 orbits) go from being attracting to neutral to repelling, and an attracting period 4 orbit is born.

Repeating this graphical process as $c \to -2$ we notice that an attracting orbit of period 2^n is born, becomes neutral and then becomes repelling and an attracting orbit of period 2^{n+1} is born.

For higher periodic orbits it is most convenient to use graphical analysis, but for period 2 orbits we can explicitly show the nature of the orbit:

Let $q_{\pm} = \frac{-1 \pm \sqrt{-4c-3}}{2}$.

- Let $c \in (\frac{-5}{4}, \frac{-3}{4})$

$P_c^{'}(x) = 2x$, $(P_c^2)^{'}(x) = 4x^3 + 4x$.

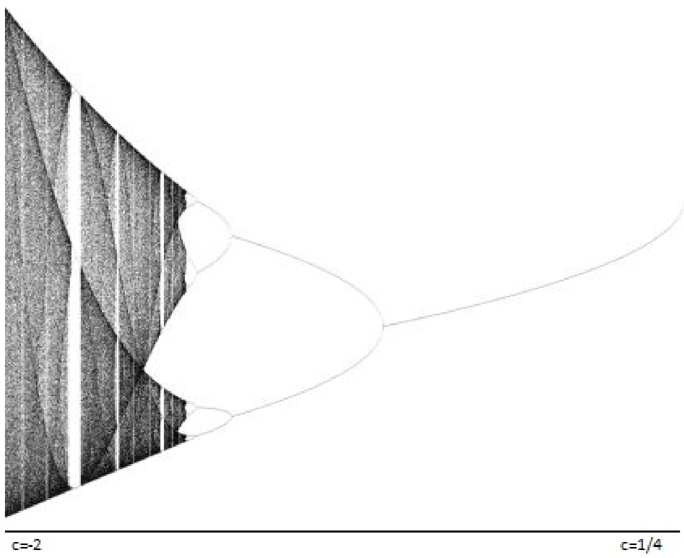
$(P_c^2)^{'}(q_{\pm}) = P_c^{'}(q_+).P_c^{'}(q_-) = (-1+\sqrt{-4c-3})(-1-\sqrt{-4c-3}) = 1 - (-4c-3) = 4c+4$.

Since $\frac{-5}{4} < c < \frac{-3}{4}$. Then $-1 < 4c+4 < 1$.

Hence $\mid (P_c^2)^{'}(q_{\pm}) \mid < 1$, so the periodic 2 orbit is attractive.

- For $c = \frac{-5}{4}$ $\mid (P_c^2)^{'}(q_{\pm}) \mid = 1$, so the orbit is neutral.
- For $c < \frac{-5}{4}$ $\mid (P_c^2)^{'}(q_{\pm}) \mid > 1$, so the orbit is repelling.

It may be reasonable to assume then that as $c \to -2$, only periodic orbits of period 2^n ($n \in \mathbb{N}$) exist. From the bifurcation diagram below, we see after the first few bifurcations, the frequency at which they occur and the orbits don't seem to follow the assumed pattern. This is known as the periodic doubling route to chaos.



Bifurcation diagram for P_c

1.4 Chaos

Chaos Theory stands as one of the crowning accomplishments within 20^{th} century science. Humans have always sought to explain the world in which they live, Chaos theory provides an explanation for some of the more confusing aspects of it. Despite this, the word “Chaos” has become overused, popularized in the mainstream and often attached to Hollywood disaster movies. One could argue that it’s catchy name is largely responsible for this, whatever the reason, it is clear that the mathematical principles of Chaos theory have been somewhat lost in translation.

One of the fundamental questions when studying a particular dynamical system is; Does this system exhibit chaotic or stable behaviour (under iteration)?

In general it is a good deal more complex than this, the nature of stability and chaos seem to be intrinsically linked. Even the seemingly simple quadratic function $F_\mu(x) = \mu x(1-x)$ (called the Logistic map) exhibit extremely complex behaviour. The naivety with which our understanding of functions such as the Logistic map was treated, was rife during the birth of this subject. Douady remarked [AD] “Whenever I told my friends I was starting with Hubbard a study of degree two polynomials in the complex plane, they would ask, “Do you expect to find anything new?””.

Definition 19. Let X be a non-empty set, and $f : X \to X$. Then f is chaotic on X if:

1. f is (topologically) transitive on X
2. Periodic points for f are dense in X
3. f has sensitive dependence on initial conditions

Note: The literature gives many different definitions for chaos, but this is the most common.

Definition 20. Given a non-empty X, the map $f : X \to X$ is topologically transitive if for any open sets U and V, $\exists n > 0$ such that $f^n(U) \bigcap V \neq \emptyset$.

This definition says then that if there exists a point $x \in X$ such that it's forward orbit is dense in X, then X is topologically transitive.

Definition 21. We say periodic points for f are dense in X if given some periodic point $x \in X$, then for any open neighbourhood of X, there are arbitrarily many other periodic points in that neighbourhood.

We say f exhibits Sensitive dependence on Initial conditions (SDIC) if for any subset $Y \subseteq X$, $\exists \varepsilon > 0$ such that for every $x \in Y$ and $\delta > 0$, $\exists y \in Y$ and $n \in \mathbb{N}$ such that $d(x, y) < \delta \implies d(f^n(x), f^n(y)) > \epsilon$.

Under this definition we have the notion of distance between iterated points, so strictly speaking we require X to be a metric space. When we study systems in the complex plane, we usually study them using the Riemann Sphere, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and to introduce the notion of distance between points diverging to infinity, we require the distance between these points to converge to zero. IE. For $p, q \in \overline{\mathbb{C}}$ if $p, q \to \infty$ (under f) then $d(p, q) \to 0$. Dynamical systems exhibit SDIC if minor changes to our starting point produce vastly different long term behaviour (under iteration).

Example 22. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z^2$. we will show that when $|z| = 1$, f exhibits SDIC.

We know $z = e^{i\theta}$. Also, $f^2(z) = e^{2i\theta} = cos(2\theta) + isin(2\theta)$

So f doubles the angle ($mod 2\pi$). Let $\theta = 2\pi t$, the $f(z) = (e^{2\pi it})^2$. Since this map will now factor out $mod 1$, f is essentially the same as the doubling map (topologically conjugate) $T(x) = 2x \, mod 1$.

Choose some point x with neighbourhood $N_\epsilon(x)$ such that $y \in N_\epsilon(x)$.

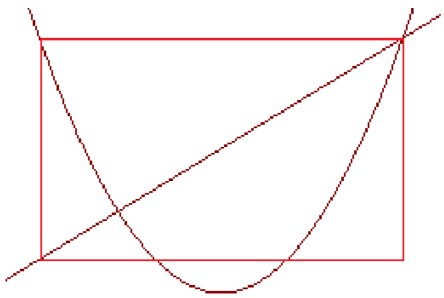
Say $d(x, y) = \delta \implies d(T(x), T(y)) = 2\delta$. So $d(T^n(x), T^n(y)) = 2^n\delta$.

So we can choose ϵ such $\epsilon > 2^n\delta$.

The distance between these points grows exponentially, and so we are done. This gives a very obvious example of SDIC, any "error" at the beginning will be compounded after iterations.

1.5 Symbolic dynamics [D1][D5]

In this section we investigate the dynamics of P_c when $c < -2$. We saw that when $c < \frac{1}{4}$, we may restrict our attention to the interval $[-p_1, p_1]$ (label this T), since this is where the dynamics of P_c is interesting, i.e. points do not just tend to infinity. Intuitively any value escapes to infinity when $c < -2$. We show that this is far from the case.



Graph of $x^2 + c$ with $c < -2$, boxed with $[-p_1, p_1]$.

Denote the region of P_c which dips outside of T by B_o. Clearly any point lying in B_0 escapes to infinity. Note that B_0 is an open set set and so $T \setminus B_0$ is comprised of two closed sets, label the left interval T_0, and the right interval by T_1. After this, we construct B_1, the region for which points lie in B_0 under one application of P_c; $B_1 = \{x \in T \mid P_c(x) \in B_0\}$. Clearly any point lying in B_1 escapes to infinity. This splits T further, $T \setminus (B_0 \cup B_1)$, and we can be sure that points which do not escape must lie in this set. Using the same idea of construction we can inductively define B_{n+1} as the region for which points lie in B_n under P_c, the region for which points lie in B_0 under P_c^{n+1}; $B_{n+1} = \{x \in T \mid P_c^{n+1}(x) \in B_0\}$. For any $x \in B_n$ $(n \in \mathbb{N})$, x will escape to the super-attracting fixed point infinity under repeated iterations.

Remark 23. It is easy to compute the number of regions which have the possibility of points which don't escape;

$$T \text{has } 1$$

$$T - B_0 \text{has } 2$$

$$(T - B_0) - B_1 \text{ has } 2^2$$

$$\vdots$$

$$(\ldots((T - B_0) - B_1)\ldots) - B_n \text{has } 2^{n+1}$$

$$\vdots$$

Because of the above, we are only interested in points in $T \setminus \bigcup_{n\in\mathbb{N}} B_n$, points which do not escape to infinity. This is set is commonly referred to as Λ.

We now prove that for sufficiently small c, Λ is a Cantor set. That is; for some $c_* < -2$, then for every $c < c_*$, Λ is a Cantor set.

Theorem 24. *For $c < c_* < -2$, Λ is a Cantor set (closed, totally disconnected and perfect)*

Proof. Λ is closed since we being with T, a closed set, and only remove open set, or label as follows;

$$T - B_0 = A_0$$

$$(T - B_0) - B_1 = A_1$$

$$\vdots$$

$$(\ldots((T - B_0) - B_1)\ldots) - B_n = A_n$$

Then each A_n ($n \in \mathbb{N}$) is a closed set and $\Lambda = \bigcap_{n\in\mathbb{N}} A_n$ is the intersection of arbitrarily many closed sets, and so is closed. Let c_* be such that for $c \le c_*$, $\mid P_c'(x) \mid > 1$ for $x \in T - B_0$ $(\star)$.[5]

Suppose now that Λ is not totally disconnected. Then there exists $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ such that $[\alpha, \beta] \subset \Lambda$. By $(\star)$ there exists $\lambda > 1$ such that $\mid P_c'(x) \mid > \lambda > 1$ for any $x \in \Lambda$. By the chain rule $\mid (P_c^n)'(x) \mid >$

[5]We need to check for $c \le c_*$, $\mid 2x \mid > 1$ for $x \in [-p_1(= \frac{-(1+\sqrt{1-4c})}{2}), -\sqrt{-p_1 - c}]$ and $x \in [\sqrt{-p_1 - c}, p_1(= \frac{(1+\sqrt{1-4c})}{2})]$. For example let $c_* = -.275$. Then $x \in [\frac{-5}{2}, -\sqrt{1.25} \approx -1.11]$ and $x \in [\sqrt{1.25} \approx 1.11, \frac{5}{2}]$. Clearly for x in these intervals $\mid 2x \mid > 1$, so we can be sure such a c_* exists.

$\lambda^n > 1$, so for any $w \in [\alpha, \beta]$; $| (P_c^n)'(w) | > \lambda^n > 1$. By the Mean Value Theorem $| (P_c^n)'(w) | = | \frac{P_c^n(\beta) - P_c^n(\alpha)}{\beta - \alpha} | > \lambda^n > 1$. $\therefore | P_c^n(\beta) - P_c^n(\alpha) | > \lambda^n | \beta - \alpha | > 1$.

From this we see the interval; is expanded by λ^n ($n \in \mathbb{N}$), this is arbitrarily large, so some point in the interval $[\alpha, \beta]$ will hit B_0 under repeated applications of P_c. But this means a point $w \in [\alpha, \beta]$ will tend to infinity under repeated applications of P_c, contradicting $w \in [\alpha, \beta] \subset \Lambda$. So Λ is totally disconnected.

A Cantor set is perfect if every point is a limit point of others points in the set, that is; Λ is perfect if every open set containing $x \in \Lambda$ contains some $y \in \Lambda$ where $y \neq x$.

Sketch of Proof that Λ is perfect: Suppose not. All end points of B_n $\to 0$ under repeated applications of P_c. Suppose all nearby points of $x \to \infty$. It can be shown under the assumption that P_c^n maps p_1 to 0 (so P_c^n has a maximum at p_1), then p_1 is eventually mapped outside of T. Contradiction.

□

Remark 25. This proof can be done without the need for c_*. We merely require that for every $x \in T - B_0$, $| (P_c^n)'(x) | > 1(\star)$ for some $n \geq 1$, and then we can apply the same proof with slightly altered details. The difficult part is making sure $(\star)$ holds, and this is why we add in the condition involving c_*.

We now move to studying the dynamical structure of Λ. For this we need Symbolic Dynamics.

First we define a space within which to operate, this will be the set of all infinite strings of 0's and 1's, and we denote it by Σ; $\Sigma = \{(a_0 a_1 a_2 \ldots) \mid a_i = 0\text{or}1\}$. We then define a metric on this space, let $a = (a_0 a_1 \ldots), b = (b_1 b_2 \ldots) \in \Sigma$. Define $e : \Sigma \times \Sigma \to \mathbb{Q}^+$ by $e(a, b) = \frac{1}{k+1}$ where k is the least index such that $a_k \neq b_k$, and $e(a, a) = 0$.

Proposition 26. *e defines a metric on Σ*

Proof. Directly from the definition we have $e(a, b) = e(b, a)$, $e(a, b) \geqslant 0$ $\forall a, b \in \Sigma$, and $e(a, b) = 0 \iff a = b$.

Let $a, b, c \in \Sigma$ and let $k_1 = \min\{i \mid a_i \neq b_i\}$ and $k_2 = \min\{i \mid b_i \neq c_i\}$. Then $e(a, c) = \frac{1}{\min\{k_1, k_2\}+1} \leqslant \frac{1}{\min\{k_1, k_2\}+1} + \frac{1}{\min\{k_1, k_2\}+1} \leqslant \frac{1}{k_1+1} + \frac{1}{k_2+1} = e(a, b) + e(b, c)$. $\therefore$ the triangle inequality is also satisfied.

□

Example 27. Let $a = (000\ldots)$ and $b = (111\ldots)$. Then $e(a, b) = 1$.

Whilst simple in it's definition (and so as mathematicians, a metric we would prefer to use), this metric will not help us in our pursuit to understand the dynamics of Λ under P_c. The reason why will become clearer later, we first define a metric which will help:

For $a, b \in \Sigma$ we define $d : \Sigma \times \Sigma \to \mathbb{Q}^+$ by $d(a, b) = \sum\limits_{i\in\mathbb{N}} \frac{|a_i - b_i|}{2^i}$.

Proposition 28. *d defines a metric on Σ*

Proof. $d(a, b) = 0 \iff | a_i - b_i |= 0 \; \forall i \iff a = b$

Clearly $d(a, b) \geqslant 0 \; \forall a, b$.

$d(a, b) = \sum\limits_{i\in\mathbb{N}} \frac{|a_i - b_i|}{2^i} = \sum\limits_{i\in\mathbb{N}} \frac{|b_i - a_i|}{2^i} = d(b, a)$

Let $a, b, c \in \Sigma$, then $d(a, c) = \sum\limits_{i\in\mathbb{N}} \frac{|a_i - c_i|}{2^i} = \sum\limits_{i\in\mathbb{N}} \frac{|a_i - b_i + b_i - c_i|}{2^i} \leqslant \sum\limits_{i\in\mathbb{N}} \frac{|a_i - b_i|}{2^i} + \sum\limits_{i\in\mathbb{N}} \frac{|b_i - c_i|}{2^i} = d(a, b) + d(b, c)$.

□

Lemma 29. *If* $d(a, b) < \frac{1}{2^{n+1}}$ *then* $d((a_1 a_{2\ldots}), (b_1 b_2 \ldots)) < \frac{1}{2^n}$

Proof. If $d(a, b) < \frac{1}{2^{n+1}}$ then $| a_i - b_i |= 0 \; \forall i = 0, 1, \ldots, n + 1$. Hence $| a_i - b_i |= 0 \; \forall i = 1, \ldots, n + 1$.

$\therefore d((a_1 a_{2\ldots}), (b_1 b_2 \ldots)) < \frac{1}{2^n}$.

□

We now prove that the map $\sigma : \Sigma \to \Sigma$ defined by $\sigma(a_0 a_1 \ldots) = (a_1 a_2 \ldots)$ is continuous. As P_c acts upon Λ, this map will give us analogous properties on Σ.

Remark 30. It becomes clear at this point why the metric e would not have helped. The above lemma plays a pivotal role in the proof that σ is continuous, and could not be replaced by e. If it were that $e(a, b) < \frac{1}{n+1}$ or $\frac{1}{2^{n+1}}$, then we exclude the possibility that a and b differ in their first digit. But σ is defined for every $a, b \in \Sigma$ including strings which differ in first digits. So if we use e in the proof that σ is continuous we will have only shown its continuity on part of Σ, not the entire set.

Proposition 31. *σ is continuous on Σ*

Proof. Let $\epsilon > 0$ and let $n > 0$ be such that $\frac{1}{2^n} < \epsilon$. We choose $\delta = \frac{1}{2^{n+1}}$.

Then if $d(a, b) = d((a_0a_1 \ldots), (b_0b_1 \ldots)) < \frac{1}{2^{n+1}} = \delta$

Then $d(\sigma(a), \sigma(b)) < \frac{1}{2^n} = \epsilon$ by the above Lemma.

So σ is continuous.

□

We now proceed to define the relationship between Λ and Σ, between P_c and σ. Define $h(x) = (s_0s_1 \ldots)$ where $s_i = 0$ if $P_c^i(x) \in T_0$ and $s_i = 1$ if $P_c^i(x) \in T_1$.

Theorem 32. *h gives a homeomorphism between Λ and Σ.*

Thus, we have a conjugacy between the dynamics in Λ under P_c and the dynamics in Σ under σ, that is $\sigma^n \circ h(x) = h \circ P_c^n(x)$. So the dynamical properties existing in Σ will also exist in Λ. With this in mind we investigate some of the dynamical properties in Σ. Firstly it is easy to see that the fixed points of Σ under σ are $(000 \ldots)$ and $(111 \ldots)$. Clearly there are no other fixed points; a fixed point $(s_0s_1 \ldots)$ mus have the property that $s_0 = s_1 = \ldots$ and since $s_i = 0$ or $s_i = 1$ this implies $(s_0s_1 \ldots)$ is either $(000 \ldots)$ or $(111 \ldots)$. So we can be sure these are the only fixed points and they correspond to the fixed points in Λ.

Periodic points also exist in Σ, for instance $a = (a_0a_1a_2 \ldots a_{n-1}a_1 \ldots)$ is a periodic point of period n, so there are 2^n periodic points of period n under σ for each $n \in \mathbb{N}$. Also note that the periodic points are dense in Σ, pick an arbitrary point $a \in \Sigma$, then the periodic point $a_n = (a_0a_1a_2 \ldots a_{n-1}a_1 \ldots)$ will come arbitrarily close to a as n increases, since by Lemma 30 we have $d(a_n, a) < \frac{1}{2^n}$. Hence $a_n \to a$ as $n \to \infty$, since a was arbitrary this shows periodic points are dense in Σ.

Remark 33. Although periodic points are dense, it is easy to construct a point in Σ which is not periodic and also not dense. Consider $a = (010011000111 \ldots)$. Under repeated applications of σ, a comes arbitrarily close to either of the fixed points (and so periodic points) but it is never actually fixed nor periodic. So it is not dense.

Astoundingly then, using the homeomorphism and dynamical properties of Σ we can convince ourselves that not only do periodic points exist in Λ, but they are in fact dense, quite contrary to the intuitive notion that all points in Λ escape to infinity.

2 Complex Dynamics

2.1 Julia Sets

We now turn our attention to the main focus of this paper, the Mandelbrot and Julia sets. To this end, we focus on a set of functions now quite familiar, the class of polynomials $P_c(z) = z^2 + c$ $(z, c \in \mathbb{C})$.

When studying the Julia sets, we almost always first meet this quadratic family. One reason for this is that the critical point is located at zero, and so from Theorem 9 it is extremely easy to locate an attracting periodic cycle (if it has one!). Later we will see this is especially useful when trying to compute the Julia set.

This section aims to give an overview of the Julia set and state some of the more interesting results surrounding it.

Definition 34. The Julia set of a function $F : \mathbb{C} \to \mathbb{C}$, is the set of points on which the function is chaotic.

The complement of this set is called the called the Stable or Fatou set. As the name implies, this is the set of points on which the function is not chaotic. The Chaotic and Stable sets, are named after the famous mathematicians Gaston Julia and Pierre Fatou, respectively. There are alternative definitions of the Julia set which may be given first in different texts, but from the definition given in this paper, we can prove the other definitions to be correct.

Alternative Julia set characterizations:

- The Julia set of $F : \mathbb{C} \to \mathbb{C}$ is the set of points under which $\{F^n\}$ fails to be a normal family for any neighbourhood of $z \in \mathbb{C}$. (This means the Fatou set is the set under which $\{F^n\}$ is a normal family for any neighbourhood of $z \in \mathbb{C}$)
- The Julia set of $F : \mathbb{C} \to \mathbb{C}$, denoted $J(F)$, is the closure of the set of repelling periodic points of F. (Where $z \in \mathbb{C}$ is a repelling periodic point if $F^n(z) = z$ for some $n \in \mathbb{N}$ and $|(F^n)'(z)| > 1$)

Proposition 35. *For any analytic $f : \mathbb{C} \to \mathbb{C}$, $J(f)$ is completely invariant.*

Proof. We delay this proof until the notion of normal families has been fully introduced.

□

Remark 36. Proposition 35 gives us an effective method for plotting Julia sets, given any point $z_0 \in J(F)$ then we can find the pre-images of z_0 and it is a fact that they are dense in $J(F)$, and so will give us an image resembling the Julia set of F.

Another characterization of the Julia set (for rational functions) is that it is the boundary between the set of points which escape to infinity under iteration, and those that are bounded.

Example 37. Consider the function $P_{-2}(z) = z^2 - 2$ we previously saw that for any $z \in \mathbb{C}\backslash[-2, 2]$, $P^n_{-2}(z) \to \infty$, hence under our definition, $J(P_{-2}) = [-2, 2]$.

We also showed previously that for $P_0(z) = z^2$ (the squaring function), the chaotic set is the set of points $z \in \mathbb{C}$ for which $|z| = 1$, and hence $J(P_0) = \{z \in \mathbb{C} \mid |z| = 1\}$. [6]

When studying the family of quadratics $P_c(z) = z^2 + c$, a natural question to ask is, what role does c have in determining if $z \in \mathbb{C}$ escapes under iteration to infinity?

It turns out we only need to consider $z \in \mathbb{C}$ with $|z| \leq |c|$, with $|c| \leq 2$.

Proposition 38. *If $|z| \geq |c| > 2$ then $|P^n_c(z)| \to \infty$.*

[6] Note that the Julia function for $f(z) = z^m$ for any $m > 1$ is the unit circle: $f^n(z) = z^{m^n} = r^{m^n} e^{m^n i\theta}$. So $| f^n(z) |=| r^{m^n} |$. Clearly if $r \in [0, 1)$ then $f^n(z) \to 0$, if $r = 1$ then $f^n(z) = 1$, and if $r \in (1, \infty)$ then $f^n(z) \to \infty$, as $n \to \infty$.

Proof. $|P_c(z)| = |z^2 + c| \geq |z|^2 - |c|$ [7]

$|z|^2 - |c| > |z|^2 - |z|$ [8]

=$|z|(\underbrace{|z| - 1}_{>1})$ [9]

$\therefore |P_c(z)| = |z^2 + c| \geq |z|^2 - |c| > |z|^2 - |z| = |z|(|z| - 1) > |z|$.

Hence, $|P_c^n(z)| \to \infty$ as $n \to \infty$.

□

The above gives a useful method for computing the Julia set P_c. We instantly know that iterations of z will tend to infinity if at any time, $|z| > max(|c|, 2)$.

Corollary 39. *Given an polynomial f with $deg(f) > 1$, $\exists R > 0$ such that if $|z| > R$ then $|f(z)| > |z|$. So $|f^n(z)| \to \infty$ if $|z| > R$.*

Example. (22 continued) We have two very obvious attracting fixed points, those being 0 and ∞. The basin of attraction of 0 is the unit of open disk, and the basin of attraction of infinity is all points z with $|z| > 1$, IE. points outside of the unit disk. Then we have a repelling fixed point at 1, for no matter how close we come to the point, say $1 + \delta$ with $\delta > 0$, the under successive iterations this point will diverge from 1. It is clear then that the boundary between these two basins of attraction gives the Julia set.

Theorem 40. *$J(P_c)$ is compact.*

Proof. We first prove that $J(P_c)$ is bounded. We have already shown that for sufficiently large z (ie. for $|z| > r = max(|c|, 2)$). Then the iterates of $z \to \infty$, so any z with $|z| > r$ cannot lie in the Julia set and so $J(P_c)$ is bounded. More formally:

Let $|z| > r$, say $z = r + \epsilon$ with $\epsilon > 0$.

$|z^2| = |P_c(z) - c| \leq |P_c(z)| + |c|$

Therefore, $|P_c(z)| \geq |z^2| - |c| \geq (r + \epsilon)|z| - |z| = |z|(r + \epsilon - 1)$

But $r + \epsilon > 2$, so $|z|(r + \epsilon - 1) > 1$

[7] {By $\triangle$ inequality $|y| = |x + y - x| \leq |x| + |y - x|$, so $|y| - |x| \leq |y - x|$. Therefore,$|y + x| \geq |y| - |-x| = |y| - |x|$}
[8] *Since* $|z| > |c|$
[9] *Since* $|z| > 2$

$P_c^n(z) \geq (r + \epsilon - 1)^n |z| \to \infty$ as $n \to \infty$

So, $J(P_c) \subseteq B_r(0)$ (Centre 0 since this is the critical point)

We now show $J(P_c)$ is closed:

First we define the filled in Julia set of P_c, commonly denoted $K(P_c)$, to be the entire set of points whose orbits do not tend to ∞

$\therefore K(P_c) = \{z \in \mathbb{C} \mid P_c^n(z) \nrightarrow \infty, n \to \infty\}$

Another characterizations of $J(P_c)$, which follows from Montel's Theorem, is $J(P_c) = \partial K(P_c)$, ie. $J(P_c)$ is the boundary of the two basins of attraction for 0 and ∞.[10]

Equipped with this definition, we show $J(P_c)$ is closed by first showing $\mathbb{C} \setminus K(P_c)$ is open:

Let $z_0 \in \mathbb{C} \setminus K(P_c)$, so $|z_0| > r = max(|c|, 2)$

So certainly for any $n \in \mathbb{N}$, $P_c^n(z_0) > r$

Therefore there exists some $\delta > 0$ such that $\forall z \in B_\delta(z_0)$, $P_c^n(z) > r$.

Hence $\mathbb{C} \setminus K(P_c)$ is open $\Longrightarrow K(P_c)$ is closed.

Note that $\mathbb{C} \setminus K(P_c) = \{z \in \mathbb{C} \mid P_c^n(z) \to \infty, n \to \infty\}$ ie. The basin of attraction of ∞,which we previously showed was an open set

Therefore $K(P_c)$ is closed and since $J(P_c) = \partial K(P_c)$, we have that $J(P_c)$ is closed.

This concludes the proof that $J(P_c)$ a compact set.

$\square$

[10] This characterisation of $J(P_c)$ only holds for rational maps. For example the family of exponential maps ce^z with $c \in \mathbb{C}$, has Julia set equal to the entire complex plane, and so it is not equal to the boundary of the filled in Julia set.

2.2 Normal Families

While it is possible to gain some insight into Julia and Fatou sets using algebraic and geometric techniques, the more complex results mostly rely on the notion of Normal families of functions, a term coined by Paul Montel in 1912.

Definition 41. A family $\mathcal{F}$ of complex analytic (possibly meromorphic) functions $\{f_n\}$ defined on some open domain U is said to be normal if every sequence of f_n's has a subsequence which converges uniformly on compact subsets of U.[11]

We say $\mathcal{F}$ fails to be normal at some $z \in \mathbb{C}$ if $\mathcal{F}$ fails to be normal in every neighbourhood of z.

We say a family $\mathcal{F}$, of meromorphic functions on a domain $U \subset \overline{\mathbb{C}}$ is equicontinuous if each $f \in \mathcal{F}$ is continuous on U. That is, given $\epsilon > 0$ there exists some $\delta > 0$ such that if $| z - z_0| < \delta$ then $| f(z) - f(z_0)| < \epsilon$ for every $z,\ z_0 \in U$ and every $f \in \mathcal{F}$.

Theorem 42. *Let $\mathcal{F}$ be a family of meromorphic functions on a compact domain $U \subset \overline{\mathbb{C}}$. Then the following are equivalent:*

1. $\mathcal{F}$ is a normal family.
2. Every sequence $(f_n)_{n=0}^{\infty}$ of maps in $\mathcal{F}$ has a (uniform) convergent subsequence.
3. $\mathcal{F}$ is equicontinuous on U.

Before we provide the proof we state the following Lemma.

Lemma 43. *Given a compact set $U \subset \overline{\mathbb{C}}$, there exists a subset $\{z_i \mid i \in \mathbb{N}\} \subset U$ which is dense in U.*

[11] Let f be a complex function. If $\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists then, then this is said to be the derivative of f at z_0. f is said to be holomorphic in some subset $U \subset \mathbb{C}$ if f is differentiable at every point in U. We say f is meromorphic in U, if there exists some closed subset $V(f) \subset U$ such that f is holomorphic in $U - V(f)$, and has poles at every point in $V(f)$.

Sketch of proof: Since U is compact it can be covered by a finite number of balls each of radius ϵ. For any $n \in \mathbb{N}$ we can choose $\epsilon = \frac{1}{n}$ (this will only effect the number of balls required). So there exists a finite number (n) of points (the centres of the balls) with distance less than $\frac{1}{n}$ from the centre of another of the balls. Then the union of these points for every $n \in \mathbb{N}$ is a dense subset in U.

Proof. [RB] (of Theorem 42) $1 \iff 2$: This follows directly from the definition

$3 \implies 2$: (This is known as the Arzela-Ascoli Theorem) Let $\epsilon > 0$. There exists some $\delta > 0$ such that if $| z - z_0| < \delta$ then $| f(z) - f(z_0)| < \epsilon$ for every $z, z_0 \in U$ and every $f \in \mathcal{F}$. Choose $A = \{z_i\}$ such that it forms a dense subset. Then $\bigcup_i B_{\frac{\delta}{2}}(z_i)$ covers U. U is compact so we can choose a finite subcover, say $\bigcup_{i=0}^{k} B_{\frac{\delta}{2}}(z_i)$. We have that $A = \{z_i\}$ form the centres of the balls, so for each $z_i \in A$ there is some $z_j \in A$ which is a distance $<\delta$ from one another, that is for each z_i, $\exists z_j \in A$ such that $| z_i - z_j |< \delta$. Choose n_0 such that $n, m \geq n_0$ and $| z_i - z_j |< \delta$, then $| f_{nn}(z_i) - f_{mm}(z_j)| < \frac{\epsilon}{3}$.

Then for any $z \in U$; $| f_{nn}(z) - f_{mm}(z)| \leq | f_{nn}(z) - f_{nn}(z_j)| + | f_{nn}(z_j) - f_{mm}(z_j)| + | f_{mm}(z_j) - f_{mm}(z)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$.

Hence we have subsequence (f_{nn}) which converges uniformly on U.

$2 \implies 3$: We have uniform convergence, so every sequence (f_n) obeys the Cauchy criterion, that is; there exists some n_0 such that if $n, m \geq n_0$ then $| f_n(z) - f_m(z)| < \frac{\epsilon}{3}$ for every $z \in U$. Since each f_r is continuous, there exists some $\delta > 0$ such that $| z_i - z_j| < \delta$ then $| f_r(z_i) - f_r(z_j)| < \frac{\epsilon}{3}$. $(\star)$[12]

We have that for any $\epsilon > 0$ when $| z_i - z_j| < \delta$ and $n, r \geq n_0$ then $| f_n(z_i) - f_n(z_i)| \leq | f_n(z_i) - f_k(z_i)| + | f_k(z_i) - f_k(z_j)| + | f_k(z_j) - f_n(z_i)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$.

Hence (f_n) is equicontinuous on U .

□

Lemma 44. Let $\mathcal{F}$ be a family of meromorphic functions on a compact domain $U \subset \overline{\mathbb{C}}$. Let $V \subset \overline{\mathbb{C}}$ be another compact domain and $g : V \to U$ be a meromorphic function. Then $\mathcal{F} \circ g$ is a normal family on V.

[12] Note the subtle difference between this and the definition of equicontinuity. We have that each f_r is continuous, so for each f_r there exists some $\delta > 0$ such that $\star$ holds. For equicontinuity we require that this δ be independent of the function we choose in $\mathcal{F}$.

Sketch of Proof: Let $f_n \in$ F and consider the sequence $(f_n \circ g) \in \mathcal{F} \circ g$. There will exists a subsequence of $f_n \circ g$ restricted to a compact subset of V which converges uniformly. This is because g restricted to this compact subset will be a compact subset of V, and so the subsequence of f_n applied to this compact set will converge uniformly.

We are now in a position to prove Proposition 35.

Proof. It will suffice to prove that the Fatou set, $F(f)$, is completely invariant, since this will imply the compliment (the Julia set) is also completely invariant. We wish to show $z \in F(f) \iff f(z) \in F(f)$. Here we only prove the $\Longleftarrow$.

($\Longleftarrow$): Let $f(z) \in F(f)$. Directly from the definition of Julia sets in terms of normal families we see that the Fatou set is the set under which $\{F^n\}$ is a normal family for any neighbourhood of $z \in \mathbb{C}$. Choose some neighbourhood U of $f(z) \in F(f)$, such that $\{f^n\}$ is normal. Choose some neighbourhood V of z such that $f(V) \subset U$. We can apply Lemma 44, showing that $\{f^n \mid n \geq 2\}$ is normal, but then $f \cup \{f^n \mid n \geq 2\}$ will also be normal since adding an extra map will not affect normality. Thus $z \in F(f)$.

□

Under our characterizations of $J(P_c)$ in terms of normal families, Montel showed that for any point in $J(P_c)$ and for any neighbourhood of that point, the family of iterates at points within that neighbourhood assume almost every point in the complex plane.

Theorem 45. *(Montels Theorem) [NS]*

Let $\mathcal{F}$ be a family of complex analytic functions on a domain U. If three values exist such that they are omitted by every $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.

IE. If there exists $a, b, c \in \overline{\mathbb{C}}$ such that $a, b, c \notin \bigcup_{f \in \mathcal{F}} f$ then $\mathcal{F}$ is normal.

We give a variant of this Theorem:

Suppose $a \neq b \in \overline{\mathbb{C}}$ such that $a, b, \notin \bigcup_{f \in \mathcal{F}} f$, then $\mathcal{F}$ is normal on U (We allow ∞ to be a limit).

Equipped with this Theorem and our definition of Julia sets in terms of normal families, this gives us an extremely important corollary:

Corollary 46. *Let f be a complex analytic map and $z_0 \in J(f)$. Let U be some neighbourhood of z_0. Then $\bigcup_{n>0} f^n(U)$ omits at most on point in $\overline{\mathbb{C}}$.*

Proof. If $f^n(U)$ omitted more than point, then by Montels Theorem, $\{f^n \mid n > 0\}$ would be a normal family, but from our definition of $J(f)$ this isn't so.

□

Definition 47. Points which are omitted by $\bigcup f^n(U)$ are called exceptional points.

Proposition 48. $\{f^n\}$ *isn't normal at any repelling periodic point.*

Proof. Let z_0 be a repelling periodic point (so it is part of a repelling periodic orbit).

Therefore, $|(f^n)'(z_0)| = \lambda > 1$, and $|(f^{nk})'(z_0)| = \lambda^k \to \infty$ as $k \to \infty$. $(\star)$

Assume $\{f^n\}$ is normal on U (some neighbourhood of z_0).

Note that $(f^{nk})(z_0) = z_0$for every k. So we can assume we do not have divergence to ∞ on U.

Since we have normality on U, there exists a subsequence $\left\{f^{nk_j}\right\}$ converging to some analytic function h on U.

So $|(f^{nk_j})'(z_0)| \to |h'(z_0)|$. But $|(f^{nk_j})'(z_0)| \to \infty$ by $(\star)$. Contradiction.

□

We now prove the characterization of Julia set in terms of Normal families holds:

Proposition 49. $J(f) = \{z \mid \{f^n\}$ *is not normal at* $z\}$

Proof. We already have that the Julia set is the closure of repelling periodic points, so the result follows from Proposition 48.

□

Proposition 50. $J(P_c)$ *has empty interior*

Proof. Suppose $J(P_c)$ does not have empty interior.

Then there exists an open subset U contained in $J(P_c)$

By our Corollary to Montels Theorem; $\bigcup_{n>0} P_c^n(U) = \overline{\mathbb{C}} - \{a\}$, where $\{a\}$ is an exceptional point.

Since $U \subseteq J(P_c)$ and J is completely invariant $\Longrightarrow \overline{\mathbb{C}} - \{a\} \subseteq J(P_c)$

But J is closed, so $J(f) = \overline{\mathbb{C}}$.

$\Longrightarrow$ The immediate basin of attraction of ∞ is in $J(P_c)$.

Contradiction.

□

Note that this doesn't hold for all functions. If f was an entire function, it could be the case that $J(f) = \overline{\mathbb{C}}$, as is the case when $f = \lambda e^z$, $\lambda \in \mathbb{C}$.

Theorem 51. $J(P_c) = J(P_c^n)$ *[NS]*

Proof. Clearly $J(P_c) \subseteq J(P_c^n)$, namely when $n = 1$.

So we wish to show $J(P_c^n) \subseteq J(P_c)$.

To this end we show that P_c and P_c^n have identical Fatou sets.

Clearly this is true though since for any polynomial f, $\{f^n\}$ is normal on some open subset U if and only if $\left\{f^{nk}\right\}$ is normal on some open subset U:

Suppose $\{f^n\}$ is normal on some open subset U, so a subsequence $\{f^{n_j}\}$ converges to some analytic function g on U.

Then $\left\{f^{n_j k}\right\} = \left\{ \underbrace{(f^{n_j}) \circ \ldots \circ (f^{n_j})}_{k \text{ times}} \right\}$ converges to g^k on U since each subsequence uniformly converges to g.

Other direction similar.

□

Proposition 52. $J(P_c) \neq \emptyset$

Proof. Suppose $J(P_c) = \emptyset$

Then $\{P_c^n\}$ is normal at every point in $\overline{\mathbb{C}}$.

Note that any point $z \in \overline{\mathbb{C}}$ is contained in a neighourhood U with centre 0, and with radius large enough so that U contains a point which diverges to ∞ under iteration, say z_1, and also contains a fixed point of P_c^n for any n, say z_2.

Note then that no subsequence $\left\{P_c^{n_j}\right\}$ can converge uniformly to an analytic function g or infinity on U since U contains both z_1 and z_2. This contradicts the fact $\{P_c^n\}$ was normal.

□

This proposition therefore shows P_c always has a repelling fixed point or a neutral fixed point (which coincides with one of our characterizations of the Julia set):

Proposition 53. *Let f be a polynomial. The repelling fixed points lie in the Julia set of f.*

Proposition 54. *$J(P_c)$ is symmetric about the origin.*

Proof. Given $z_0 \in J(P_c)$ we wish to show $-z_0 \in J(P_c)$.

We rely on the invariance of $J(P_c)$:

Given $z_0 \in J(P_c)$, $(z_0^2 + c) \in J(P_c)$.

IE. $P_c(z_0) \in J(P_c)$.

The $P_c^{-1}(P_c(z_0)) = \{-z_0, z_0\} \in J(P_c)$ since $J(P_c)$ is completely invariant.

□

2.3 The Mandelbrot Set

This section will deal with the Mandelbrot set, one of the most aesthetical, complicated sets within mathematics.

Despite the many advances within the study of complex dynamics in the 20^{th} century, it wasn't until the utilization of powerful computers, that the mathematical community, and the world, first glimpsed upon its profound beauty.

It is the bifurcation set, or the "indexing" set of P_c, and lives in the c-complex plane.

Definition 55. The Mandelbrot set is the set of $c's$ such that $P_c^n(0) \nrightarrow \infty$ as $n \to \infty$. That is, $\mathcal{M} = \{c \mid P_c^n(0) \nrightarrow \infty\}$.

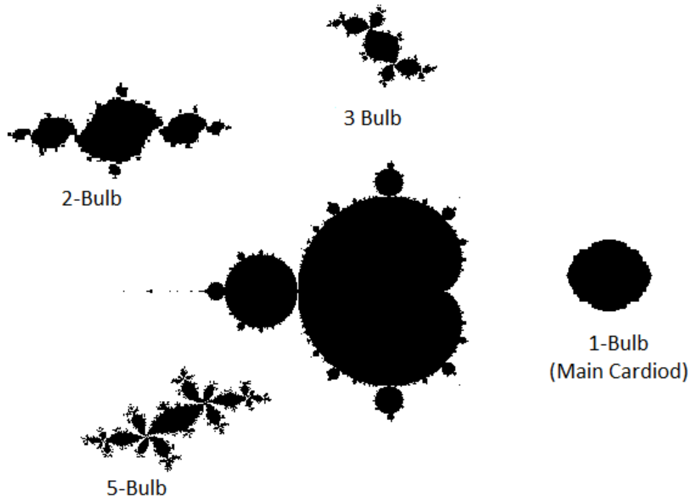
This shows the importance of the critical point 0, and it follows from the next Theorem that the Mandelbrot set is the set of $c's$ such that K_c is connected.

Theorem 56. *1. If $P_c^n(0) \to \infty$ as $n \to \infty$, then $K(P_c) = J(P_c)$, and is totally disconnected (a Cantor Set).*

2. If $P_c^n(0) \nrightarrow \infty$ as $n \to \infty$, then $J(P_c)$ is a connected set.

Theorem 56 gives another way to characterise the Mandelbrot set, $\mathcal{M} = \{c \in \mathbb{C} \mid J(P_c)$ is connected$\}$.

Astonishingly the Julia set for P_c can only have one piece, or an infinite number of pieces. It was this dichotomy that lead Mandelbrot to investigate Julia sets. Mandelbrots own definition at the time was $\mathcal{M} = \{c \mid c \to c^2 + c \to (c^2 + c)^2 + c \to \ldots \nrightarrow \infty\}$, which gives a very stark visual relationship between the Julia sets and the critical orbit.



The Mandelbrot set with associated Julia sets.

In general, classifying and understanding dynamical systems is extremely difficult and there are not many successful cases (when studying a system of degree> 2 we are usually totally lost). The dynamics of the P_c is one of the rare cases when we can fully explain the dynamics of the system, and the Mandelbrot set gives us this understanding.

So we have:

- Julia sets - Depending on the orbits of $z \in \mathbb{C}$
- Mandelbrot set - Depending on $c \in \mathbb{C}$, classifying Julia sets.

We now give some of the basic properties of the Mandelbrot set.

Theorem 57. *$\mathcal{M}$ is simply connected*

Proof. We show in a future section that $\overline{\mathbb{C}} \setminus \mathcal{M} \cong \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Since $\overline{\mathbb{C}} \setminus \mathcal{M}$ is simply connected, its compliment $\mathcal{M}$ is simply connected.

□

Note: It is still unknown at the time of writing this paper if $\mathcal{M}$ is locally connected.[13]

[13][LCS] We say that X is locally connected at x if for every open set V containing x there exists a connected, open set U with . The space X is said to be locally connected if it is locally connected at x for all x in X.

[JH] Whilst Theorem 57 seems rather trivial, it was conjectured by Mandelbrot himself that $\mathcal{M}$ was not connected, when he sent his paper to the printer with $\mathcal{M}$ drawn in it, the printer thought the "islands" were ink dots and promptly deleted them, Mandelbrot had to redraw them on in pencil. It was eventually proven to be connected by Douady and Hubbard (along with many other important results in complex dynamics).

Proposition 58. *If* $|c| > 2$*, then* $c \notin \mathcal{M}$

(From Proposition 38, we see $\mathcal{M}$ is contained in the disk of radius 2, centred at 0)

Proof. We wish to show that $P_c^n(0) \to \infty$ as $n \to \infty$ for $|c| > 2$. We prove this by induction on n.

$|P_c(0)| = c$,

$|P_c^2(0)| = |c^2 + c| \geq |c|^2 - |c| = |c|(|c| - 1) > |c| > 2$,

$|P_c^3(0)| = |(P_c^2(0))^2 + c| \geq |P_c^2(0)|^2 - |c| = |P_c^2(0)||P_c^2(0)| - |c| \underbrace{>}_{\text{using } |P_c^2(0)|>|c|} |P_c^2(0)||c| - |P_c^2(0)| = |P_c^2(0)|(|c| - 1) \underbrace{\geq}_{\text{using } |P_c^2(0)|>|c|(|c|-1)} |c|(|c| - 1)^2$.

Suppose $|P_c^m(0)| \geq |c|(|c| - 1)^{m-1}$ holds $\forall m \leq n$.

Then $|P_c^{n+1}(0)| = |(P_c^n(0))^2 + c| \geq |P_c^n(0)|^2 - |c| = |P_c^n(0)||P_c^n(0)| - |c| > |P_c^n(0)|(|c| - 1) \underset{\text{By Inductive Hypothesis}}{\geq} |c|(|c| - 1)^{n-1}(|c| - 1) = |c|(|c| - 1)^n$.

Therefore $|P_c^n(0)| \to \infty$ as $n \to \infty$. So $c \notin \mathcal{M}$.

□

Proposition 59. $\mathcal{M} \bigcap \mathbb{R} = [-2, \frac{1}{4}]$

Proof. Let $c \in \mathcal{M} \bigcap \mathbb{R}$, so c is real and $P_c^n(0) \nrightarrow \infty$ as $n \to \infty$.

We wish to show that only for $c \in [-2, \frac{1}{4}]$ is the critical orbit bounded.

We solve $P_c(x) = x$.

Therefore, $x = \frac{1 \pm \sqrt{1-4c}}{2}$, if:

$c > \frac{1}{4}$, then x is not real.

$c = \frac{1}{4}$, $x = \frac{1}{2}$.

$c < \frac{1}{4}$, x has 2 real roots.

If $c > \frac{1}{4}$, the $|P_c^n(0)|$ is unbounded. Suppose it weren't, IE. $c \to c^2+c \to (c^2+c)^2+c \to \ldots \to a$ for some $a \in \mathbb{R}$.

Then this $a \in \mathbb{R}$ would satisfy $a^2 + c = a \Longrightarrow c = a(1-a)$.

So $c = a$ or $c = 1-a$, contradiction since a cannot be real if $c > \frac{1}{4}$.

Therefore $P_c^n(0) \to \infty$ as $n \to \infty$ if $c > \frac{1}{4}$.

For $c \in \mathcal{M} \bigcap \mathbb{R}$, $c \leq \frac{1}{4}$.

We have an upper bound, but by previous proposition, $|c| \leq 2$ if $c \in \mathcal{M}$, so we have a lower bound, IE. $c \geq -2$.

Therefore $c \in [-2, \frac{1}{4}]$.

□

Why is the critical orbit important?

Julia and Fatou sets are determined by the location of attracting cycles and from Theorem 9, the critical orbit is contained in an attracting cycle and so provides a natural starting point.

An obvious question when studying non-linear (IE. polynomials of degree> 1) complex dynamics is why do we restrict ourselves to P_c? Aside from the location of the critical point, it is because any general complex quadratic is topologically conjugate to P_c.

Let $F = az^2 + bz + d$ with $a, b, d \in \mathbb{C}$. We need a map $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $P_c \circ G = G \circ F$ for some $c \in \mathbb{C}$

IE. $P_c = G \circ F \circ G^{-1}$

Let $G(z) = az + \frac{b}{2}$, G is a homeomorphism since it is clearly continuous and has inverse $G^{-1}(z) = \frac{1}{a}(z - \frac{b}{2})$.

$P_c \circ G(z) = P_c(az + \frac{b}{2}) = (az + \frac{b}{2})^2 + c = a^2z^2 + abz + \frac{b^2}{4} + c$.

Similarly,

$G \circ F(z) = G(az^2 + bz + d) = a(az^2 + bz + d) + \frac{b}{2} = a^2z^2 + abz + ad + \frac{b}{2}$.

If we let $c = ad + \frac{b}{2} - \frac{b^2}{4}$, then F is conjugate to P_c.

2.4 Analysis of the Mandelbrot Set

2.4.1 Periods within the Mandelbrot Set

Now that we have a bound on the Mandelbrot Set it is natural to ask, do we have bounds within bounds? That is, do all points behave the same within the Mandelbrot Set?

$\mathcal{M}$ consists of a main cardioid, and infinitely many buds attached to it. Attached to each of these buds are countably infinite many buds, ad infinitum.

The main cardioid is the set of c's under which P_c has an attracting fixed point. So there exists a fixed point in the main cardioid and every point within the main cardioid will converge to this fixed point after repeated iterations.

It follows each bud consists of $c-$values for which P_c has an attracting periodic cycles of some period $k = 1, 2, 3, \ldots$.

For the main cardioid this is simple $P_c(z) = z$, i.e. $z^2 + c = z$.[14]

We also know that there is an attracting fixed orbit, so we must have $|P_c'(z)| = |2z| < 1$

Of course, switching $<$ with $=$ will give the boundary of c-values for P_c has an attracting fixed point.

We solve the equations:

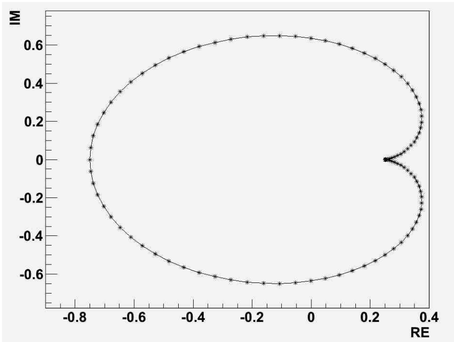
$|2z| = 1$ (1)

$z^2 + c = z$ (2)

[14]This attracting fixed point is unique: Suppose a, b are fixed points of $P_c(z) - z = 0$, we have $a + b = 1$ and $ab = c$. Also $P_c'(a) + P_c'(b) = 2a + 2b = 2$. So a or $b \geq 1$, and hence P_c has one attracting fixed point.

From (1) we obtain $z^2 = \frac{1}{4}$.

Let $z = \frac{1}{2}e^{i\theta}$ with $0 \leq \theta < 2\pi$. Then $z^2 = \frac{1}{4}e^{2i\theta} = \frac{1}{4}e^{4\pi i}$ so (1) is satisfied, and $c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}$.



Main cardioid of $\mathcal{M}$

The 2-bulb:

- $P_c^2(z) = (z^2 + c)^2 + c = z$ (1)
- $|(P_c^2)'(z)| = |4z(z^2 + c)| = 1$ (2)

From (1), $z^4 + 2cz^2 + c^2 + c = z$.

$\therefore z^4 + 2cz^2 - z + c(c + 1) = 0$.

$\therefore (z^2 - z + c)(z^2 + z + c + 1) = 0$.

So we have $c = z - z^2$ or $c = -1 - z - z^2$.

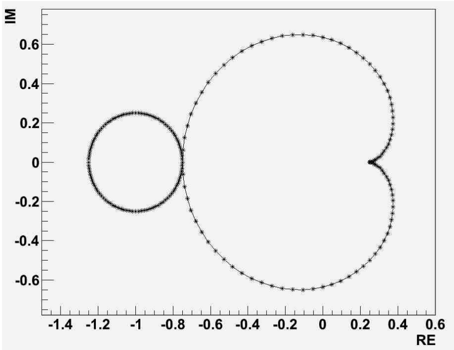
$c = z - z^2$ is the solution for the main cardioid, and so is also a solution for the period 2 bulb. (Since any fixed point of $P_c$is also fixed under P_c^2)

We also have $c = -1 - z - z^2$. (3)

From (2) and (3) , $|4z(z^2 + c)| = |4z(z^2 - 1 - z - z^2)| = | - 4z(1 + z)| = |4(-z - z^2)| = |4(c + 1)| = 1$.

Therefore $|c + 1| = \frac{1}{4}$

The boundary of the 2-bulb is given by a circle of radius $\frac{1}{4}$, centred at -1.



Main cardioid and 2-bulb of $\mathcal{M}$

(Similarly for the bud with period $k$we solve $P_c^k(z) = z$and $|(P_c^k)'(z)| = 1$for c.)

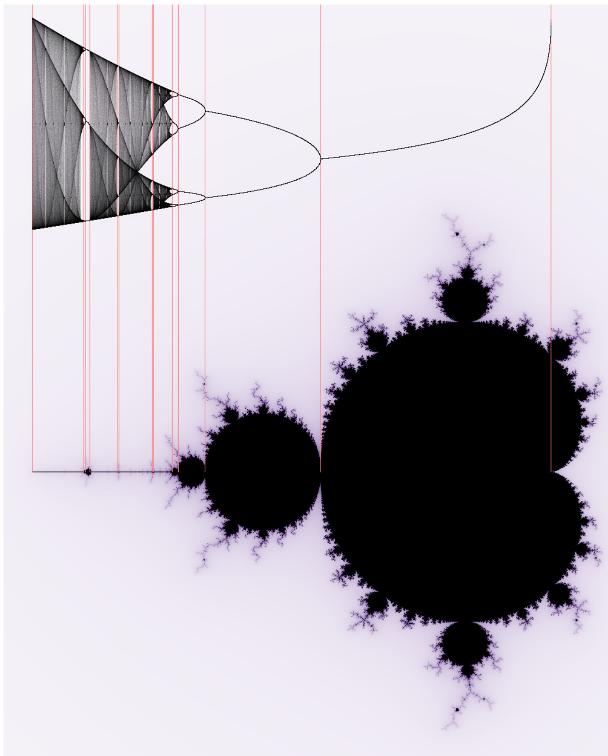
As in the real quadratic case, as c decreases, P_c becomes chaotic. We describe the dynamics as c decreases through the main cardioid, and into the $2-$bulb:

When c is in the main cardioid, P_c has two fixed points, $a = \frac{1-\sqrt{1-4c}}{2}$ is attracting, whilst $b = \frac{1+\sqrt{1-4c}}{2}$ is repelling. P_c also has a repelling period 2 cycle $\{\alpha, \beta\}$:

These are solutions the two solutions to $z^2 + z + c + 1 = 0$. They are repelling since $|(P_c)'(a).(P_c)'(b)| = |(-1 + \sqrt{1-4(c+1)}).(-1 - \sqrt{1-4(c+1)})| = |4(c+1)|$.

$|4(c+1)| > 1$ exactly when $c > \frac{-3}{4}$. That is, when c is in the main cardioid.

As $c \to \frac{-3}{4}$, $a, \alpha, \beta \to \frac{-1}{2}$ and at $c = \frac{-3}{4}$ we have $a = \alpha = \beta$ and they give a neutral fixed point of P_c. As c decreases through this value, a becomes repelling and $\{\alpha, \beta\}$ becomes an attracting 2 cycle, this is easily checked. The process of a repelling cycle becoming neutral and eventually attractive (and then repelling again) is repeated for higher periodic orbits as c continues to decrease, and is commonly referred to as period doubling route to chaos.



This image shows the correspondence between Mandelbrot set, and the Bifurcation diagram associated with it. We see that as c approaches -2, bifurcations become more rapid, and eventually the set becomes chaotic.

In general, primary bulbs (those attached to the main cardioid) meet the main cardioid when $c = \frac{1}{2}e^{2\pi i\theta} - \frac{1}{4}e^{4\pi i\theta}$ where $\theta = \frac{p}{q}$, and this bulb will have period q (sometimes then the bulb is called the $\frac{p}{q}$-bulb). Let us check this for the period 2 bulb:

$\theta = \frac{1}{2}$, $c = \frac{1}{2}e^{\pi i} - \frac{1}{4}e^{2\pi i\theta} = \frac{-1}{2} - \frac{1}{4} = \frac{-3}{4}$ which is indeed the intersection of the 2 bulb with the main cardioid.

- Pictorial evidence shows for any $\frac{p}{q}$-bulb, an antenna emanates from it containing q branches. Notice the similarity between this and the number of components the Julia set will have when c is in the $\frac{p}{q}$-bulb. Also note the q components are separated if a fixed point is removed from the Julia set.
- For any $\frac{p}{q}$-bulb, the associated bulbs attached to it all have period kq and the larger bulbs have minimal period.
- The Julia sets for which θ is irrational are a good deal more complicated and contain Siegel discs, named after the mathematician Carl Ludwig Siegel. Siegel discs appear when the function in question is conjugate to an irration rotation of the complex unit disc.[CS]

2.5 Green's function and External Rays

To further analyse the Mandelbrot set we introduce the notion of external rays and Green's function. These results are attributed to Douady and Hubbard, who produced some groundbreaking papers concerning complex dynamics during the early 1980's. Douady himself explained his ideas as follows: [AD]

"Imagine a capacitor made of an aluminum bar shaped in such a way that its cross-section is $\mathcal{M}$; placed along the axis of a hollow metallic cylinder. Set the bar at potential zero and the cylinder at a high potential. This creates an electric field in the region between the cylinder and the bar. An electric potential function is also established in this region. Assume that the radius of the cylinder is large (with respect to the chosen unit), that its height is large compared to its radius and that the length of the bar is equal to this height. We restrict our attention to the plane perpendicular to the axis of the cylinder, through its middle. In this plane, the electric potential defines equipotential lines enclosing the set $\mathcal{M}$ (which is the cross section of the bar). Following the electric field, one gets field-lines, called the external rays of $\mathcal{M}$: Each external ray starts at a point x on the boundary of $\mathcal{M}$; and reaches a point y of the great circle which is the cross section of the cylinder (practically at infinity). The position of y is identified by an angle, called the external argument of x with respect to the $\mathcal{M}$ set."

Definition 60. [EV1] The Green's function of K_c, $G_c : \mathbb{C} - K_c \to \mathbb{R}$ is given by $G_c(z) = \lim_{n\to\infty} 2^{-n} \log |P_c^n(z)|$.

From this definition we see G_c has some interesting properties:

$G_c(P_c(z)) = \lim_{n\to\infty} 2^{-n} \log |P_c^n(P_c(z))| = \lim_{n\to\infty} 2^{-n} \log |P_c^{2n}(z)| = 2. \lim_{n\to\infty} 2^{-n} \log |P_c^n(z)| = 2.G_c(z)$. $(\star)$

From $(\star)$ if we let U be some circle, then $G_c(P_c^{-n}(U)) = \frac{G_c(U)}{2^n} = k$. (some constant)

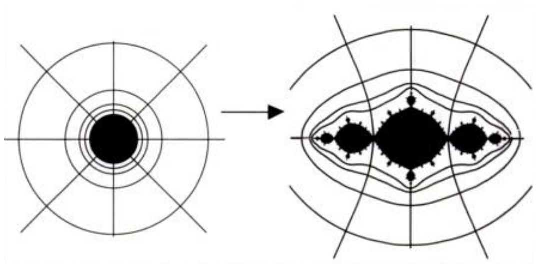
Of course altering the number of iterations will give different constants.

So every pre-image of U gives an equipotential curve around the Julia set. Lines orthogonal to the equipotential curves are called external rays.

The work of Douady and Hubbard led to external rays, which proved to be an invaluable tool when studying intricate, complicated sets like the Mandelbrot set.

We work within the confines of the Riemann sphere, so the point ∞ is defined and is a fixed point of P_c.

[D3] We then define an isomorphism in some neighbourhood of ∞. Let $c \in \overline{\mathbb{C}}$, then there exists a neighbourhood of ∞, say U_c in $\overline{\mathbb{C}}$ and an isomorphism $\phi_c : U_c \to V_r$ where $V_r = \{z \in \mathbb{C} \mid |z| > r\}$. We may extend this to give $\phi_c : \overline{\mathbb{C}} \setminus K_c \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ such that $\phi_c(P_c(z)) = (\phi_c(z))^2$, i.e. P_c is conjugated to P_0.



[PJS] The correspondence between external rays on the unit disk and external rays emanating from a filled in Julia set ($c = -1$).

Note that this map is defined for $|z| > max\,\{|c|, 2\}$, and ϕ_c is given by $\phi_c(z) = \lim\limits_{n\to\infty} (P_c^n(z))^{\frac{1}{2^n}}$

Remark. The above construction is usually referred to as the Bottcher isomorphism.

Definition 61. The external ray of argument θ is given by $\mathcal{R}_{(c,\theta)}(t) = \phi_c^{-1}(te^{2\pi i\theta})$

$= \{z \mid arg(\phi_c(z)) = constant\}$.

Denote all rays with angle θ to be $\mathcal{R}_{(c,\theta)}$, $\theta \in [0, 2\pi)$.

Note the connection between G_c and ϕ_c:

$G_c(z) = log|\phi_c(z)|$ for $z \in \overline{\mathbb{C}} \setminus K_c$.

It is a fact that if the Julia set is connected then each external ray of argument θ lands on the Julia set.

2.5.1 External rays on $\mathcal{M}$

Define $\Phi : \overline{\mathbb{C}} \setminus \mathcal{M} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ by $\Phi(c) = \phi_c(c)$.

Note that $\phi_c(c)$ is always defined since $c \in U_c$.

Douady and Hubbard managed to show that Φ is an isomorphism, much like the Bottcher Isomorphism $\big(\overline{\mathbb{C}} \setminus K_c \cong \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}\big)$, $\big(\overline{\mathbb{C}} \setminus \mathcal{M} \cong \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}\big)$.

Obviously it can be deduced from this that the Mandelbrot set is connected. We have the analogous definition of external rays for $\mathcal{M}$; $\mathcal{R}_\theta(t) = \Phi^{-1}(te^{2\pi i\theta})$. Again, it can be shown that for all $\mathcal{R}_\theta$, with $\theta \in [0, 2\pi)$, lands at a unique point in $\partial\mathcal{M}$, providing $\mathcal{M}$ is locally connected. This proviso has yet to be proved or disproved at the time of writing this paper. We now turn our attention to the external rays which land on $\mathcal{M}$, and what this tells us about the dynamics of $\mathcal{M}$.

2.5.2 External rays with rational arguments

Douady and Hubbard formulated and consequently proved their famous landing theorem which says all external rays with rational arguments do land on $\mathcal{M}$, and the dynamics of $\mathcal{M}$ at these points are determined by $\theta = \frac{p}{q}$ with $\gcd(p, q) = 1$.

Theorem 62. *[D3] Let $\theta = \frac{p}{q}$ with $\gcd(p, q) = 1$. Then the external ray of angle θ lands on $\mathcal{M}$, at some point c_θ.*

1. If q is odd, then c_θ is a root point hyperbolic component of $\mathcal{M}$. [15]
2. If q is even, then c_θ is a Misiurewicz point. [16]

[15] Hyperbolic components of $\mathcal{M}$are components for which P_c has an attracting cycle. It is still unknown at the time of writing this paper if the Hyperbolic components of $\mathcal{M}$ equals the interior of $\mathcal{M}$. c_θ is a root of a component if it is a point at which a smaller bulb is attached to a larger one, or it is the cusp of the Mandelbrot set (or even a mini Mandelbrot set).

[16] A point c is Misiurewicz point if the critical orbit of P_c is strictly preperiodic. That is the orbit of the critical point leads to a periodic cycle not containing the critical point. Clearly Misiurewicz points are completely contained in $\mathcal{M}$ since if the critical orbit is preperiodic then it is bounded.

Theorem 63. *[D2] Suppose a bulb B consists of c-values for which P_c has an attracting q-cycle. Then the root point of this bulb is the landing point of exactly 2 external rays, and the angles of these rays have period q under doubling.*

The rays determine the ordering of bulbs in the Mandelbrot set. We only have one bulb with an attracting cycle of period 2, this is the $\frac{1}{2}$-bulb. So the rays that land at the root point of this bulb must have period 2 under the function $T : [0,1) \to [0,1)$ given by $T(\theta) = 2\theta \; mod 1$. The only two rays which have period 2 under T are $\frac{1}{3}$ and $\frac{2}{3}$, so these are the external rays which land at the $\frac{1}{2}$-bulb.

Consider the $\frac{1}{3}$-bulb, the rays that land at the root point of this bulb must have period 3 under T. The only rays which have period 3 under T are $\left\{\frac{1}{7}, \frac{2}{7}, \ldots, \frac{6}{7}\right\}$. The only two rays which between 0 and $\frac{1}{3}$ with this property are $\frac{1}{7}$ and $\frac{2}{7}$. So these are the rays landing at the $\frac{1}{3}$-bulb. (Of course, the other rays $\frac{3}{7}, \ldots, \frac{6}{7}$ land at other primary bulbs with period 3)

Consider the $\frac{1}{4}$-bulb, the rays that land at the root point of this bulb must have period 4 under T. The only rays which have period 4 under T are $\left\{\frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{4}{15}, \frac{6}{15}, \frac{7}{15}, \frac{8}{15}, \frac{9}{15}, \frac{11}{15}, \frac{12}{15}, \frac{13}{15}, \frac{14}{15}\right\}$. The only two rays which between 0 and $\frac{1}{7}$ with this property are $\frac{1}{15}$ and $\frac{2}{15}$. So these are the rays landing at the $\frac{1}{4}$-bulb. (Of course, the other rays $\frac{3}{15}, \ldots, \frac{14}{15}$ land at other primary bulbs with period 4)

Remark 64. If θ is periodic, say with period k. Then c_θ is the root of a hyperbolic component with period k. We come to the relation $c_{\theta_1} = c_{\theta_2} \iff \theta_1 \sim \theta_2 \iff p_1 q_2 = p_2 q_1$.

We now present algorithms for computing equivalence classes of θ.

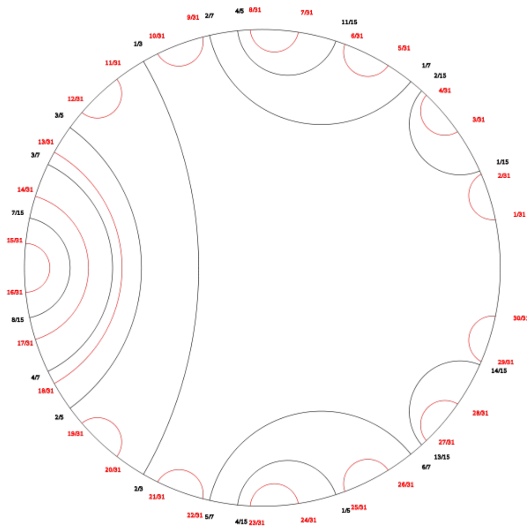
2.6 Lavaurs Theorem

[BB] Let $\theta \in [0,1)$ be a point on $e^{2\pi i\theta}$. If $\theta_1 \sim \theta_2$ connect them by a geodesic in $\mathbb{D}$(that is, arcs of circles and diameters in $\mathbb{D}$[17] that meet $\partial\mathbb{D}$ orthogonally). Then proceed as follows:

[17] The Poincaré disk model.

1. Connect $\frac{1}{3}$ and $\frac{2}{3}$ by a geodesic.
2. Suppose all points of period l for $2 \leq l < k$ have been connected. Points corresponding to rationals of period k are connected as follows:

- All arcs are disjoint.
- If $\theta_1$is the smallest rational in $[0, 1)$ of period k not yet connected, connect θ_1 to the next smallest $\theta_2 > \theta_1$, observing the point above.



[AM]

This then describes an equivalence relation on the set of rational numbers with odd denominator in $[0, 1]$. If we were to collapse every line to a point, we would have a set homeomorphic to the Mandelbrot set, and we can see what components the root point is between.

The set we obtain from this algorithm is often called the abstract Mandelbrot set, and is topologically equivalent to the Mandelbrot set.

Notice that this algorithm makes use of the doubling map $mod1$ on the circle to identify periodic orbits.

For instance $T(\frac{1}{3}) = \frac{2}{3}$ and $T(\frac{2}{3}) = \frac{1}{3}$ and so the line which we join to these two points corresponds to a 2 cycle.

Fact. *[BB] Periodic point of period k are of the form $\frac{n}{2^k-1}$ where n is a natural number between 0 and $2^k - 2$.*

Example 65.

Period of Bulb	Cycles
1	$0 \to 0$
2	$\frac{1}{3} \to \frac{2}{3} \to \frac{1}{3}$
3	$\frac{1}{7} \to \frac{2}{7} \to \frac{4}{7} \to \frac{1}{7}, \frac{3}{7} \to \frac{6}{7} \to \frac{5}{7} \to \frac{3}{7}$
4	$\frac{1}{15} \to \frac{2}{15} \to \frac{4}{15} \to \frac{8}{15} \to \frac{1}{15}, \frac{3}{15} \to \frac{6}{15} \to \frac{12}{15} \to \frac{9}{15} \to \frac{3}{15}, \frac{7}{15} \to \frac{14}{15} \to \frac{13}{15} \to \frac{11}{15} \to \frac{7}{15}$
$\vdots$	$\vdots$

Proposition 66. *$\theta = \frac{p}{q}$ is periodic if and only if q is odd.*

Proof. We know periodic points are of the form $\frac{n}{2^k-1}$, so we must be able to write $\frac{p}{q}$ in this form, that is q must divide $2^k - 1$ for some k. Clearly if q is even then it cannot divide $2^k - 1$ for any k, since this is an odd number. So q is odd.

Now suppose q is odd, then q and 2 are coprime. It follows from Euler's theorem that $2^{\phi(q)} \cong 1(modq) \Longrightarrow q \mid 2^{\phi(q)} - 1$.

□

2.7 Rotation Numbers in the Mandelbrot Set

We saw in section $2.4.1$ that each primary bulb (bulbs attached to the main cardioid) has an associated number q, which is the period of the attracting cycle within that bulb. We also associated each primary bulb with a rational number $\frac{p}{q}$. We shall now fully explain what this number is, that is, what p we associate to a period q primary bulb.

This can be done by simply looking at the bulb in question or by the associated Julia set. We present a method for finding p by looking at the primary bulb.

[MB] As previously commented, a primary bulb of period q has q emanating from the intersection point of the main antenna (which is attached to the bulb). We then find the shortest antenna and associate a number p to it. This is done by associating each antenna with a counter-clockwise revolution $\frac{r}{q}$ with $r \in \{0, 1, \ldots, q-1\}$, and reading off the shortest antenna and letting $r = p$. Note that we cannot use Euclidean distance as a measurement here, we must use a rather ad hoc method for associating the length to antennae, we will say more on this in future sections, and give a method for measuring portions of the Mandelbrot set.

Example 67. Consider the $\frac{2}{5}$-bulb. We calculate $p = 2$, by looking at Fig $2.7.1$ and determining how many counter-clockwise revolutions from the emanating antenna (where $\frac{1}{5}$ turn will bring us to the antenna to the right of the one we are looking at) we require to bring us to the shortest antenna. From Fig $2.7.1$, we see the shortest antenna is located $\frac{2}{5}$ turns from the principal spoke.

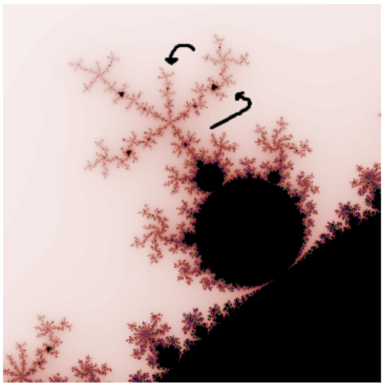


Fig $2.7.1$

So we associate a rational number $\frac{p}{q}$ with a primary bulb of period q since the attracting periodic orbit will rotate $\frac{p}{q}$ revolutions about a central fixed point with each iteration.

Example 68. We could also have used the associated Julia set (from a point inside the $\frac{2}{5}$-bulb) to determine this number, and instead of using an intuitive notion of length of antenna, we use an intuitive notion of size of components. Again we use the same idea, of counter-clockwise turns. From Fig $2.7.2$ we see the smallest component of the Julia set is located $\frac{2}{5}$ turns from the principal component of the Julia set.

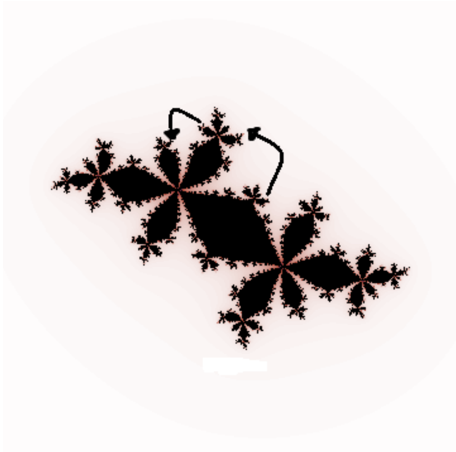


Fig $2.7.2$

So every bulb attached to the main cardioid is associated to a rational number $\frac{p}{q} \in (0,1)$. Fig $2.7.3$ gives these numbers for a few of the principal bulbs.

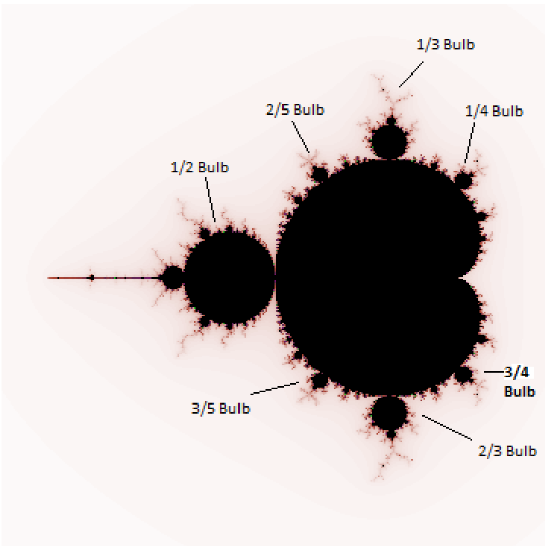


Fig $2.7.3$

2.8 Farey Addition

So far we have seen that each primary bulb admits an attracting orbit of period q, and this period is constant over that particular bulb. We have also seen that we associate a rotation number $\frac{p}{q}$ with each primary bulb. Incredibly, we can read this number directly off the bulb we are looking at, that is, the dynamical information contained in each primary bulb is directly linked to the geometry of the bulb. We now discuss Farey addition, which is used to determine the largest bulb in between two primary bulbs, and find the rotation number of that bulb.

Definition 69. The Farey sequence of order $n \in \mathbb{N}^+$ is the sequence of ascending rationals in $[0, 1]$ with denominator less than or equal to n. The set of all Farey sequences is often called the Farey tree (all rationals in $[0, 1]$).

We compute Farey sequences inductively and make use of Farey addition, which goes as follows: [D2]

Given two rationals $\frac{p}{q}, \frac{r}{s} \in [0, 1]$, $\frac{p}{q} \oplus \frac{r}{s} = \frac{p \oplus r}{q \oplus s}$.

We begin with $\frac{0}{1}$ and $\frac{1}{1}$ which is the Farey sequence of order 1.

$\frac{0}{1} \oplus \frac{1}{1} = \frac{1}{2}$, so the Farey sequence of order 2 consists of $\frac{0}{1}$, $\frac{1}{2}$ (called the Farey child of $\frac{0}{1}$ and $\frac{1}{1}$) and $\frac{1}{1}$.

Two rationals $\frac{p}{q}, \frac{r}{s} \in [0, 1]$ are called Farey neighbours if $ps - qr = \pm 1$. We compute the Farey child by using Farey addition on Farey neighbours. So to obtain the Farey sequence of order 3 we would use Farey addition to add $\frac{0}{1} \oplus \frac{1}{2} = \frac{1}{3}$, $\frac{1}{2} \oplus \frac{1}{1} = \frac{2}{3}$, and obtain $\left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$.

We continue in this way for every $n \in \mathbb{N}^+$ to give every rational in $[0, 1]$.

Theorem 70. *Given two primary bulbs with rotation numbers $\frac{p}{q}$ and $\frac{r}{s}$. The primary bulb with the largest attracting orbit in between these two bulbs has rotation number $\frac{p}{q} \oplus \frac{r}{s}$.*

Remark 71*.* As previously mentioned, the bulbs with largest cycles are the "largest" bulbs, giving another example of how dynamical information is contained within the geometry of the Mandelbrot set.

Example 72. Given the $\frac{1}{2}-$bulb and $\frac{1}{3}-$bulb, the largest bulb between these two has rotation number $\frac{2}{5}$ (and so has period 5).

It is clear then that we can associate every primary bulb to a rational number in $[0, 1]$, and if we were to shrink each bulb down to a point, these two sets would be isomorphic, as previously alluded to in our study of the abstract Mandelbrot set.

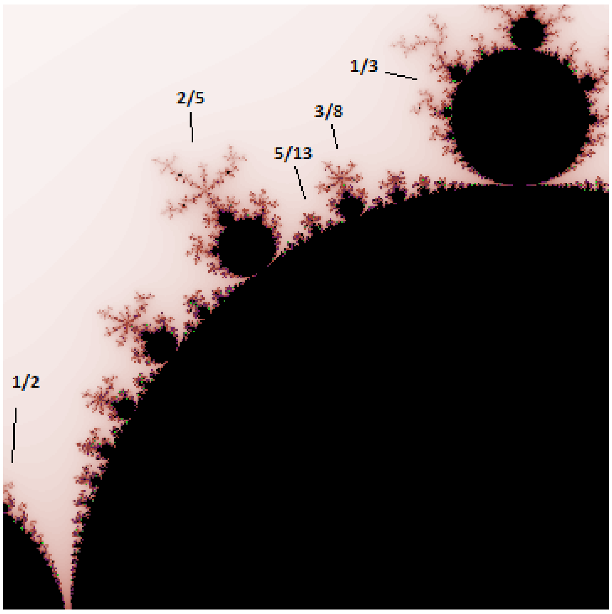
2.8.1 The Fibonacci Sequence

The Mandelbrot set possesses many interesting properties, but one property that few expected is the appearance of Fibonacci numbers within the set:

As we saw in 2.8 the largest bulb between the $\frac{1}{2}-$bulb and $\frac{1}{3}-$bulb is the $\frac{2}{5}-$bulb. The largest bulb between the $\frac{1}{3}-$bulb and $\frac{2}{5}-$bulb is the $\frac{3}{8}-$bulb.

Continuing in this way we obtain the sequence $\frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13}, \frac{8}{21}, \ldots$

Clearly both the numerator and denominator give the Fibonacci sequence.



2.9 Encirclements of the J_c and $\mathcal{M}$

[PJS][CT] We turn our attention to equipotentials in the Julia set once again. In previous sections we made precise what these were but did not provide a decent geometrical interpretation of the potentials. That is left for this section.

Sequential encirclements of J_c and $\mathcal{M}$ provide better and better approximations of the sets. They decompose $A(\infty)$ in to distinct subset which diverge to ∞ at the same rate.

Note that if $|z_n|$ ever exceeds $r = max\{c, 2\}$ then $P_c^n(z) \to \infty$ as $n \to \infty$.

The initial approximation of J_c is simply the disc of radius $r = max\{c, 2\}$, centre 0.

Notation: $S_c^{(0)} = \{z_0 \mid |z_0| \leq r\}$.

Now allow one iteration of points remaining (points satisfying $|z_0| \leq r$), we then obtain a better approximation of J_c:

$S_c^{(-1)} = \{z_0 \mid |z_1| \leq r\}$.

We repeat this *ad inifitum:* $S_c^{(-n)} = \{z_0 \mid |z_n| \leq r\}$ $n = 0, 1, 2, \ldots$

It is clear that $S_c^{(-n)} \to J_c$ as $n \to \infty$.

When $c = 0$, then $S_c^{(0)}$ is a disc of radius 2.

- $S_c^{(-1)} = \{z_0 \mid |z_1| \leq 2\} = \{z_0 \mid |z_0^2| \leq 2\} = \{z_0 \mid |z_0| \leq \sqrt{2}\}$

⋮

- $S_c^{(-n)} = \left\{z_0 \mid |z_0| \leq 2^{\frac{1}{2^n}}\right\}$

This can be done for any value of c.

So we have that the encirclements are approximations of J_c, and the larger n is, the better the approximation is. Since each encirclement is “smaller” than the next we can also write $S_c^{(-n)} = \left\{z_0 \mid z_n \in S_c^{(0)}\right\}$.

We can also look at forward images of $S_c^{(0)}$; $S_c^{(n)} = \left\{z_n \mid z_0 \in S_c^{(0)}\right\}$. We do not in general, need to limit ourselves to looking at $S_c^{(0)}$, we can set an arbitrary target set T: $S_c^{(n)}(T) = \{z_n \mid z_0 \in T\}$. When $c = 0$, $S_c^{(n)}(T)$ $\left(\forall n \in \mathbb{Z} \text{ with } S_c^{(0)} = T\right)$ gives discs bounded by equipotential lines given by $D^{(n)} = \{z \mid log_2|z| \leq 2^n\}$.

Equipotential lines which encircle the Julia sets will in general, not be perfect discs (this only happens for J_c when $c = 0$) but rather deformed discs which converge and diverge from J_c depending on the value of c (that is, depending upon the shape of the Julia set). Although for encirclements "far away" from J_c do resemble perfectly circular discs.

The distortion on the encirclements is visually apparent, they often converge and diverge from J_c in an irregular fashion (Fig $2.9.1$), this causes a problem since we stated that encirclements provide better approximations for the Julia set, but this defect seems unrelated to the Julia set.

The reason we have this problem is the choice of the target set T.

Note. $J_c^{(n)} = \lim_{n\to\infty} S_c^{(n-l)}(D^{(l)})$ where $D^{(l)} = T$, converges to a set with an equipotential boundary.

We have $J_c^{(k)} = \left\{ z \mid \lim_{l\to\infty} \frac{log_2|z_l|}{2^l} \leq 2^k \right\}$ has an equipotential boundary, with potential function $p_c(z_0) = \lim_{l\to\infty} \frac{log_2|z_l|}{2^l}$. This potential has a big advantage, it allows us to drop the bound r, giving a better target set, and provides encirclements of J_c whilst also giving equipotentials (r bound does not do this).

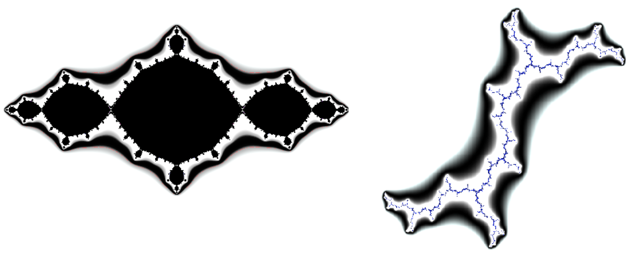


Fig $2.9.1$ Encirclements of the Julia set for $c = -1$ and $c = -i$.

2.9.1 Encirclements of $\mathcal{M}$ [PJS]

Define encirclements of $\mathcal{M}$ analogously to J_c. $\mathcal{M}_k = \left\{ c \mid \lim_{l\to\infty} \frac{log_2|z_l|}{2^l} \leq 2^k,\ z_0 = c \right\}$.

How do we arrive at $\mathcal{M}_k$? We know that $\mathcal{M} \subset$ Disc of radius 2. Just as before, define a target set T with radius≥ 2. Set $R^{(-k)}(T) = \{c \mid z_k \in T,\ z_0 = c\}$ with $z_0, z_1, z_{2,\ldots}$ the critical orbit.

Note if $z_k \in T$ then so are $z_0, z_1, z_{2,\ldots}, z_{k-1}$. Indeed let r be the radius of the Target set T (we choose T to be a disc) and consider $\mid c \mid \geqslant r$. Let z_n be a point in the orbit of $c = z_0$ with $\mid z_n \mid \geqslant \mid c \mid$. Then $\mid z_{n+1} \mid \geqslant \mid z_n \mid^2 - \mid c \mid \geqslant \mid z_n \mid (\mid z_n \mid -1) \geqslant \mid z_n \mid (r-1) \geqslant \mid z_n \mid$. We can take z_n

to be z_0, so we have $r \leqslant\mid c \mid\leqslant\mid z_0 \mid\leqslant\mid z_1 \mid\leqslant \ldots$ It follows then that if $z_k \in T$ so are $z_0, z_1, z_{2,\ldots}, z_{k-1}$.

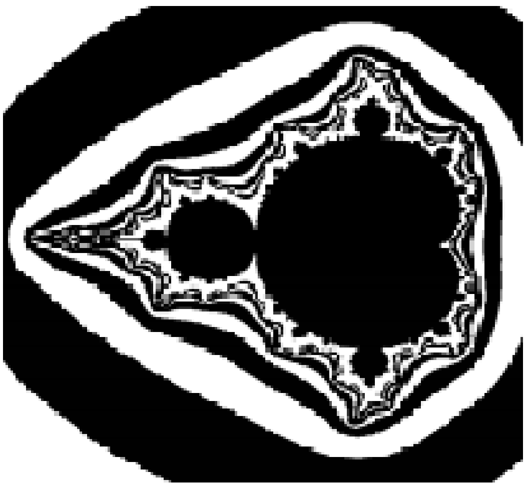
That is, $R^{(-k)}(T)$ are the values of c for which the first k iterations of $z_0 = c$ stay within the target set T. Note that $\mathcal{M} \subset T = R^{(0)}(T)$.

$R^{(-1)}(T)$ is the set of c's for which $z_0 = c$ and $z_1 = c^2 + c$ are both in T. So $R^{(-1)}(T) \subset R^{(0)}(T)$, clearly $\mathcal{M} \subset R^{(-1)}(T)$. By induction then, $\mathcal{M} \subset \ldots \subset R^{(-k)}(T) \subset \ldots \subset R^{(0)}(T) = T$ and $\bigcap\limits_{0 \leq k < \infty} R^{(-k)}(T) = \mathcal{M}$.

There is an important difference between encirclements of $\mathcal{M}$ and K_c:

$R^{(-k)}(T)$ is not an image of $R^{(-k-1)}(T)$. Obviously $R^{(-k)}(T)$ depends on T, and so we need to define T in such a way that we get the required encirclements around $\mathcal{M}$. We use the same target set as we did for J_c, let $D^{(l)} = \left\{z \mid log_2|z| \leq 2^l\right\}$ $(l = 0, 1, \ldots)$ as T, and define $\mathcal{M}_k$ as $\lim\limits_{l \to \infty} R^{(k-l)}(D^{(l)})$ for any k.

We can again define $\mathcal{M}_k$ analogously to $J_c^{(k)}$, for c belongs to $R^{(k-l)}(D^{(l)})$ if $log_2|z_{l-k}| \leq 2^l$ or $\frac{log_2|z_{l-k}|}{2^{l-k}} \leq 2^k$. Letting $l \to \infty$ will have the same effect as $l - k \to \infty$. So $\mathcal{M}_k = \left\{c \mid \lim\limits_{l \to \infty} \frac{log_2|z_l|}{2^l} \leq 2^k,\ z_0 = c\right\}$. There a subtle differences between $\mathcal{M}_k$ and $J_c^{(k)}$, for $J_c^{(k)}$ we keep c fixed and work with $z \to z^2 + c$ whereas for $\mathcal{M}_k$ we vary $c \in \mathbb{C}$. As with J_c, the equipotential and external rays of $\mathcal{M}$ have a one to one correspondence between the equipotential and external rays of the unit disc.



[CT] Encirclements of $\mathcal{M}$.

We have only discussed encirclements when J_c is connected, but it can be done when J_c is also disconnected, if we have that $S_c^{(0)}, S_c^{(-1)}, S_c^{(-2)}, \ldots S_c^{(-k)}$

are completely connected curves which encircle J_c, and deform the complex plane in to two distinct sets, yet $S_c^{(-k-1)}$ does not, it can be shown J_c is disconnected, but successive encirclements still provide an approximation of this Julia set.

2.10 Binary Expansion

Lavaurs theorem gave us a method to determine which rays land on $\mathcal{M}$. We now work towards measuring the size of parts of the Mandelbrot set using external rays. To do this we will need the notion of binary expansion of external rays

We make use of the doubling function mod1 mentioned in Lavaurs Theorem, that is $T : [0,1) \to [0,1)$ given by $T(\theta) = 2\theta mod 1$. First, partition $[0,1)$ into two distinct subset $I_0 = [0, \frac{1}{2})$ and $I_1 = [\frac{1}{2}, 1)$. For each $\theta \in [0,1)$ we associate a binary string $\theta_0\theta_1\theta_2\ldots$ under the condition $\theta_k = \{ {0 \text{ if } T^i(\theta)\in I_0 \atop 1 \text{ if } T^i(\theta)\in I_1}$. This binary string gives the binary expansion of θ.

Example 73. Let $\theta = \frac{2}{3}$. Then $\theta \in I_1$, $T(\theta) \in I_0$ and $T^2(\theta) = \theta$. Hence the binary expansion of θ is $\overline{10}$. Note this makes sense since under the doubling function T we have $\frac{2}{3} \to \frac{1}{3} \to \frac{2}{3} \to \ldots$ (both $\frac{1}{3}$ and $\frac{2}{3}$ have period 2)

2.11 Limbs

In light of Theorem 62, we can define the notion of a limb of the Mandelbrot set. We know two rays land on a point $c \in \mathcal{M}$. Define the limb of $\mathcal{M}$ with respect to this c to be the component of the Mandelbrot set which would be cut off from the main cardioid when deleting this particular c.

Before we give a method for computing in size of limbs of the Mandelbrot set we must give a method to determine the angle of rays landing at a root point of a $\frac{p}{q}$ bulb. As we know, two rays will land at the root point, so we will have two different angles, we label them $s_{-}(\frac{p}{q})$ and $s_{+}(\frac{p}{q})$, where $s_{-}(\frac{p}{q}) < s_{+}(\frac{p}{q})$.

We now give Devaney's algorithm for computing these angles [MB]:

First let $R_{\frac{p}{q}}(\theta) = e^{2\pi i(\frac{p}{q}+\theta)}$ denote the rotation of $\frac{p}{q}$ revolutions around the unit circle. Much like in section 2.10, we split the unit circle in to two distinct subsets, $I_0^- = \left\{\theta \mid 0 < \theta \leq 1 - \frac{p}{q}\right\}$ and $I_1^- = \left\{\theta \mid 1 - \frac{p}{q} < \theta \leq 1\right\}$, and call this the lower partition. We also split the unit circle again, $I_0^+ = \left\{\theta \mid 0 \leq \theta < 1 - \frac{p}{q}\right\}$ and $I_1^+ = \left\{\theta \mid 1 - \frac{p}{q} \leq \theta < 1\right\}$, and call this the upper partition.

Define to itineraries $\overline{s_{\pm}(\frac{p}{q})}$ relative to $R_{\frac{p}{q}}$, where $\overline{s_{-}(\frac{p}{q})}$ is called the lower itinerary and is associated with the lower partition, and $\overline{s_{+}(\frac{p}{q})}$ is called the upper itinerary and is associated with the upper partition. $\overline{s_{\pm}(\frac{p}{q})}$ is defined as follows:

$$s_{\pm}(\tfrac{p}{q}) = s_1 s_2 \ldots s_q \text{ [18] where } s_i == \begin{cases} 0 \text{if} & R^{i-1}_{\frac{p}{q}}(\frac{p}{q}) \in I_0^{\pm} \\ 1 \text{if} & R^{i-1}_{\frac{p}{q}}(\frac{p}{q}) \in I_1^{\pm} \end{cases}$$

Theorem 74. *[D2] The two rays landing at a root point c of the $\frac{p}{q}$ bulb are given by $s_{\pm}(\frac{p}{q})$.*

Example 75. Consider $\frac{p}{q} = \frac{1}{3}$, then $I_0^- = (0, \frac{2}{3}]$, $I_1^- = (\frac{2}{3}, 1]$, $I_0^+ = [0, \frac{2}{3})$, and $I_1^+ = [\frac{2}{3}, 1)$. The orbit of $\frac{1}{3}$ under $R_{\frac{1}{3}}(\frac{1}{3})$ is as follows:

$\frac{1}{3} \to \frac{2}{3} \to 1 \to \frac{1}{3} \to \ldots$ since $R_{\frac{1}{3}}(\frac{1}{3})$ corresponds to a rotating the unit circle around its centre by angle $2\pi.\frac{2}{3}$.

Now $\frac{1}{3}$ lies in I_0^- and I_0^+, $\frac{2}{3}$ lies in I_0^- and I_1^+ and 1 lies in I_1^- and I_0^+. Thus $s_{-}(\frac{1}{3}) = \overline{001}$ and $s_{+}(\frac{1}{3}) = \overline{010}$.

We can revert back to find $s_{-}(\frac{1}{3})$ and $s_{+}(\frac{1}{3})$ in base 10. $\overline{001} = \sum\limits_{1 \leq n < \infty} \frac{1}{2^{3n}} = \frac{1}{2^3}(\frac{1}{1-\frac{1}{2^3}}) = \frac{1}{2^3}(\frac{2^3}{2^3-1}) = \frac{1}{7}$ and $\overline{010} = \sum\limits_{1 \leq n < \infty} \frac{1}{2^{2n}} = \frac{1}{2^2}(\frac{1}{1-\frac{1}{2^3}}) = \frac{1}{2^2}(\frac{2^3}{2^3-1}) = \frac{2}{7}$.

Now we define the length of the $\frac{p}{q}$ limb to the length of $[s_{-}(\frac{p}{q}), s_{+}(\frac{p}{q})]$. This makes sense since the size of the limb will therefore be related to

[18] Since we are working in the $\frac{p}{q}$ bulb, we know that the bulb has period q. consequently $\overline{s_{\pm}(\frac{p}{q})}$ is an infinite repeating sequence of q $0's$ and $1's$.

the number of external rays approaching the limb. Also note that this length will never be zero since $s_-(\frac{p}{q}) \neq s_+(\frac{p}{q})$, this is because these two binary sequences will always differ by the last two digits (modulo period).

Theorem 76. *[D2] The size of the $\frac{p}{q}$ limb is $\frac{1}{2^q-1}$.*

Proof. $\overline{s_+(\frac{p}{q})} - \overline{s_-(\frac{p}{q})}$ will only differ in their last two digits.

$\overline{s_+(\frac{p}{q})} = \sum_{1 \leq n < \infty} \frac{1}{2^{nq-1}}$ and $\overline{s_-(\frac{p}{q})} = \sum_{1 \leq n < \infty} \frac{1}{2^{nq}}$.

$\overline{s_+(\frac{p}{q})} - \overline{s_-(\frac{p}{q})} = \left(2 \sum_{1 \leq n < \infty} \frac{1}{2^{nq}}\right) - \left(\sum_{1 \leq n < \infty} \frac{1}{2^{nq}}\right) = \sum_{1 \leq n < \infty} \frac{1}{2^{nq}} = \frac{1}{2^q}\left(\frac{1}{1-\frac{1}{2^q}}\right) = \frac{1}{2^q-1}$

□

This precise definition of length agrees with the intuitive notion we may have when viewing various limbs.

Example 77. From Fig$2.10.1$ we intuitively expect the $\frac{2}{3}$-limb to be larger than the $\frac{3}{4}$-limb, and indeed this is the case. The size of $\frac{2}{3}$-limb is $\frac{1}{2^3-1} = \frac{1}{7}$ and the size of the $\frac{3}{4}$-limb is $\frac{1}{2^4-1} = \frac{1}{15}$.

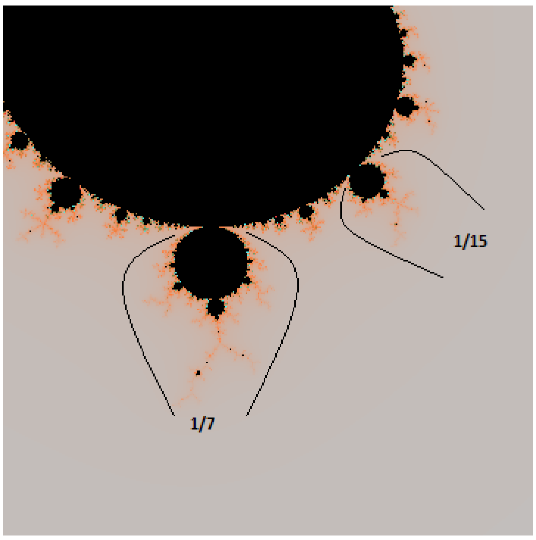
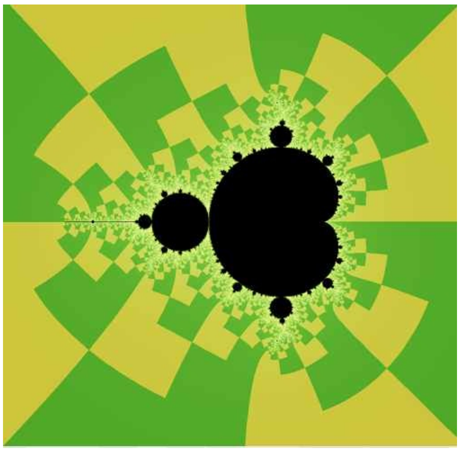


Fig $2.10.1$

2.11.1 Schleicher's Algorithm

Example 78. The method of referring to external rays in binary digits has it's advantages pictorially too. For example, we see from Fig

$2.11.1$, the blocks of colours refer to digits 0 and 1 and as we converge to $\mathcal{M}$ if we cross colours, we change digits. For instance we do not change colour as we approach the cusp of the main cardioid, so the external ray is $\overline{0} = \overline{1}$, but when approaching the $\frac{1}{2}$-bulb (from above) we repeatedly change colours, and so this corresponds to the ray $\overline{10}$, whereas approaching it from below, the same thing happens, but with colours interchanged, so this corresponds to the ray $\overline{01}$.



[MN] Fig $2.11.1$

Finally, we give an algorithm in the same vein as Devaney's to find the rays which land at the $\frac{p}{q}$ bulb. This algorithm gives a method for determining the the rays which land on bulbs attached to the main cardioid of $\mathcal{M}$. A side benefit of this algorithm is that it allows us to determine the rays which land at root points of bulbs attached to these bulbs, and so forth. The algorithm makes use of Farey addition, which we defined in section 2.8, and goes as follows:

[D3] We first begin with the $\overline{0}$ external ray, which land on the cusp of the main cardioid, and the two external rays $\overline{10}$ and $\overline{01}$, which land at the root of the $\frac{1}{2}$-bulb. From this we find the external rays which land at the root of the largest $\frac{p}{q}$-bulb lying between the $\overline{0}$ external ray and the root point of the $\frac{1}{2}$-bulb. This is of course the $\frac{1}{3}$-bulb. We find the largest bulb between the $\frac{0}{1}$ and $\frac{1}{2}$ bulb by farey addition, so this is the $\frac{0}{1} \oplus \frac{1}{2} = \frac{1}{3}$-bulb. We then determine the rays landing at this bulb as follows, first note the closest two external rays already constructed are $\overline{0}$ (landing at the cusp of the $\frac{0}{1}$-bulb) and $\overline{01}$ (one of the two rays landing at the root point of the $\frac{1}{2}$-bulb). To get the ray closest to the cusp of the main cardioid, we write 0, and then we write 01, giving $\overline{001}$. Similarly to get the ray closest to the ray closest to the root of the $\frac{1}{2}$-bulb, we write 01, and then 0, giving $\overline{010}$. We continue in this way to compute all external rays landing at root points of the $\frac{p}{q}$-bulbs attached to the main cardioid:

1. Suppose we know the external rays landing at the $\frac{p}{q}$-bulb and the $\frac{r}{s}$-bulb. We compute largest bulb between them using Farey addition; $\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$-bulb.
2. Find the ray closest to the $\frac{p+r}{q+s}$-bulb, say $\overline{s_1 \dots s_n}$ (landing at the $\frac{p}{q}$-bulb) and $\overline{t_1 \dots t_m}$ (landing at the $\frac{r}{s}$-bulb).
3. The ray closest to the $\frac{p}{q}$-bulb is given by $\overline{s_1 \dots s_n t_1 \dots t_m}$ and the ray closest to the $\frac{r}{s}$-bulb is given by $\overline{t_1 \dots t_m s_1 \dots s_n}$.

Using this algorithm we can compute every ray landing at every root point of bulbs attached to the main cardioid. We can also determine the rays which land at root points of bulbs attached to these bulbs, and so forth. It follows from the fact that for any hyperbolic component of the Mandelbrot set, there exists a homeomorphism taking the main cardioid to this component, and taking primary bulbs of the main cardioid to primary bulbs of the component, secondary bulbs to secondary bulbs, and so forth. A hyperbolic component of $\mathcal{M}$ is an interior component which consists of c-values for which P_c has an attracting periodic cycle. It is conjectured that all interior components of $\mathcal{M}$ are hyperbolic, but certainly all primary bulbs attached to the main cardioid are. Because of this we can use an altered version of the Schleicher algorithm to compute external rays landing at primary bulbs of a given $\frac{p}{q}$-bulb attached to the main cardioid.

1. Suppose we know the external rays landing at the $\frac{p}{q}$-bulb, say $\overline{s_1 \dots s_n}$ and $\overline{t_1 \dots t_m}$. We first find the rays attached to the unique period doubling bulb attached to the $\frac{p}{q}$-bulb (Note that this is the largest bulb attached to the $\frac{p}{q}$-bulb). The ray landing at the root point of the period doubling bulb closest to $\overline{s_1 \dots s_n}$ is given by $\overline{s_1 \dots s_n t_1 \dots t_m}$, and the ray closest to $\overline{t_1 \dots t_m}$ is given by $\overline{t_1 \dots t_m s_1 \dots s_n}$.
2. We then find the largest bulb between the root of the $\frac{p}{q}$-bulb and the root of the period doubling bulb using Farey addition.
3. The ray closest to the $\frac{p}{q}$-bulb is given by $\overline{s_1 \dots s_n s_1 \dots s_n t_1 \dots t_m}$ and the ray closest to the period doubling bulb is given by $\overline{s_1 \dots s_n t_1 \dots t_m s_1 \dots s_n}$.

Continuing in this way gives all external rays attached to the $\frac{p}{q}$-bulb.

3 Conclusion and possible areas for further study.

There are areas of study which we could further investigate; The similarity between Julia and Mandelbrot sets at Misiurewicz points is particularly interesting. This result was first discovered by the mathematician [TL] Tan Lei and states that at any Misiurewicz point $c \in \mathcal{M}$, $\mathcal{M}$ and J_c are asymptotically self similar, that is, they look the same after rotation and scaling. Tan Lei's theorem gives an astounding correspondence between $\mathcal{M}$ and J_c. Since there are infinite number of Misiurewicz points and at these points, $\mathcal{M}$ is the "same" as J_c, $\mathcal{M}$ offers a visual map for an infinite number of Julia sets. It is known that there also an infinite number of mini-Mandelbrot sets contained within $\mathcal{M}$. The existence of an infinite number of mini-Mandelbrots seems to contradict Tan Lei's result, since Julia sets cannot contain mini-Mandelbrots (J_c is supposed to be self similar, but it is not if it contains a mini-Mandelbrot). So how can $\mathcal{M}$ and J_c be asymptotically self similar at an infinite number of points? [EV2] It turns out that as we magnify $\mathcal{M}$ by a factor of $\lambda \in \mathbb{C}$, the mini-Mandelbrots shrink by a factor of λ^2. The size of mini-Mandelbrots decrease faster than the window in which we view the Mandelbrot set, and so the existence of mini-Mandelbrots is a non-issue.

It was the aim of this paper to provide an introduction to quadratic polynomial dynamics and an overview of the Mandelbrot and Julia sets, a rare success story in the field of non-linear dynamics.

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